An Introduction to the Lagrange and Markov Spectra through the Lens of Generalized Markov Numbers

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Abstract. This text gives a self-contained introduction to the Lagrange and Markov spectra through the lens of generalized Markov numbers. Generalized Markov numbers are the integer quantities obtained from generalized Markov equations arising in a generalized cluster-algebraic setting. They extend the role of the classical Markov numbers, which appear in the discrete part of the Lagrange and Markov spectra, and they come equipped with their own combinatorial and matrix-theoretic structures.

The aim of this text is to explain how this generalized Markov-number viewpoint provides a new way to organize the tools used in the study of the Lagrange and Markov spectra. The early chapters develop the classical background in a self-contained manner. We recall continued fractions, quadratic irrationals, binary quadratic forms, and bi-infinite continued-fraction sequences, and use them to define and study the Lagrange and Markov constants. This prepares the language in which the generalized theory can be compared with the classical one.

The later chapters introduce generalized Markov equations and the structures attached to generalized Markov numbers. These include generalized Markov trees, fraction labels, fence posets, generalized Markov distances, generalized Cohn matrices, and generalized strongly admissible sequences. Using these tools, we describe generalized discrete Markov spectra and explain how their values are realized as Lagrange constants of quadratic irrationals and as Markov constants of binary quadratic forms with rational coefficients.

In the generalized Markov-number framework developed in this text, the classical role of Markov numbers is recovered as the classical case of a broader theory. The text revisits the traditional connection between Markov numbers and the Lagrange and Markov spectra from this more general viewpoint.

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CHAPTER 1

Background and Organization of the Text

The Lagrange and Markov spectra enter number theory through two elementary-looking questions. One concerns the approximation of real numbers by rational numbers, and the other concerns the values of indefinite binary quadratic forms on integral points. The definitions are simple, but the resulting spectra are far from elementary. Already at the first stage, one is led to continued fractions, quadratic irrationals, binary quadratic forms, and two-sided infinite sequences.

This text studies these spectra through the lens of generalized Markov numbers. The role of this opening chapter is to explain the background needed for that viewpoint and to indicate how the rest of the text is organized. Rather than assuming that the reader is already familiar with the classical theory in detail, we use this chapter as a guide to the prerequisite material: which notions are needed, why they are introduced, and how they later become connected with generalized Markov numbers.

1. History of the Lagrange and Markov Spectra and Generalized Markov Numbers

We begin with a brief historical overview of the objects studied in this text and of the surrounding theory.

1.1. Emergence of Continued Fractions and Diophantine Approximation. Before discussing the Lagrange and Markov spectra, let us review the part of Diophantine approximation theory that underlies them. Roughly speaking, Diophantine approximation asks how well an irrational number α can be approximated by rational numbers $\frac{p}{q}$. There are many possible meanings of "how well," but the most elementary question is the following.

PROBLEM 1.1.1. *Let α be an irrational number. For every $\varepsilon > 0$, does there always exist a rational number $\frac{p}{q}$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \varepsilon?$$

Yes. From the modern construction of the real numbers this is immediate, and in fact infinitely many such rational numbers exist. It is nevertheless natural to go one step further and ask the following question.

PROBLEM 1.1.2. *How can one construct a sequence of rational numbers with good approximation properties that converges to an irrational number α?*

One answer is provided by the sequence of rational numbers obtained by truncating the continued-fraction expansion. This point of view goes back to Euler's 1737 paper [**Eul37**]. The relation between irrational numbers and infinite continued fractions discovered by Euler is now understood as the following correspondence. This does not mean, however, that Euler himself proved the theorem in this modern form.

THEOREM 1.1.3. *Let $\mathscr{S}$ be the set of all infinite sequences whose first entry is an integer and whose subsequent entries are positive integers. Then the map*

$$F : \mathscr{S} \to \mathbb{R} \setminus \mathbb{Q}, \quad (a_k)_{k=0}^{\infty} \mapsto [a_0; a_1, a_2, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

is a bijection.

Lagrange also used continued fractions in his 1770 paper [**Lag70**] to characterize quadratic irrationals, that is, irrational numbers that occur as roots of quadratic equations with rational coefficients.

THEOREM 1.1.4 (Lagrange's Theorem). *The continued-fraction expansion of an irrational number α is eventually periodic if and only if α is a quadratic irrational.*

The theory of continued fractions developed during this period later became a central tool in Diophantine approximation.

In the nineteenth century, the basic question of how well irrational numbers can be approximated by rational numbers came to be studied in the following quantitative form.

PROBLEM 1.1.5. *For an irrational number α, how large can one take $L > 0$ and $n > 0$ so that there exist infinitely many rational numbers $\frac{p}{q}$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^n}?$$

Let us examine this problem through a concrete example. For example, consider the irrational number $\pi = 3.141592\ldots$. Rational numbers very close to π exist no matter how small an error tolerance we impose. For example, rational numbers satisfying $\left|\pi - \frac{p}{q}\right| < \frac{1}{1000}$ can be produced as

$$\frac{p}{q} = \frac{3141}{1000}, \frac{6283}{2000}\left(=\frac{31415}{10000}\right), \frac{314159}{100000}, \frac{392699}{125000}\left(=\frac{3141592}{1000000}\right), \frac{15707963}{5000000}\left(=\frac{31415926}{10000000}\right), \ldots. \tag{1.1.1}$$

However, all of these rational numbers have relatively large denominators. In general, the smaller the required error is, the larger the denominator of a rational number satisfying it must be. Fractions with small denominator are sparse on the number line, and hence are less likely to lie close to a specified irrational number.

Thus, in rational approximation, one must control not only the error but also the size of the denominator. The "smallness of the denominator relative to the approximation error" is measured by the parameters n and L when the error bound is written in the form $\frac{1}{Lq^n}$. Since the scale is difficult to interpret if both L and n are allowed to vary simultaneously, one usually fixes one of them and studies the other.

First fix $L = 1$ and consider the supremum of the possible values of n for the rational numbers appearing in (1.1.1). This value is computed as $n = -\frac{\log\left|\pi - \frac{p}{q}\right|}{\log q}$, and for the fractions displayed above it is approximately $1.076, 1.222, 1.115, 1.213, 1.085$, from left to right. On the other hand, the same computation for $\frac{22}{7}$ and $\frac{355}{113}$ gives approximately 3.429 and 3.201, respectively, which are much larger. This means that $\frac{22}{7}$ and $\frac{355}{113}$ give much better approximations than one would expect from the size of their denominators.

If an approximation for which L or n can be taken large is called a good approximation, then Problem 1.1.5 asks the following: for a given irrational number α, how far can we raise the parameters L and n measuring the quality of approximation before rational approximations of that quality cease to exist infinitely often?

The meaning of this question is not yet completely clear. What property of an irrational number is being measured? To clarify this, we first ask what it means for a number to have few good rational approximations. The following fact is fundamental.

THEOREM 1.1.6. *For any rational number α, the supremum of the real numbers n for which there exist infinitely many reduced fractions $\frac{p}{q}$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

is 1.

Let us compare this with Dirichlet's theorem [**Dir42**], one of the starting points of Diophantine approximation theory.

Theorem 1.1.7 (Dirichlet's Theorem). *For any irrational number α, the supremum of the real numbers n for which there exist infinitely many rational numbers $\frac{p}{q}$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

is at least 2*.*

These two theorems show that rational numbers themselves are the real numbers with the fewest good rational approximations. Equivalently, when $L = 1$ is fixed, the supremum of the possible exponents n can be regarded as a measure of how far a real number is, arithmetically, from being rational.

This supremum is called the *irrationality exponent* and is denoted by $\mu(\alpha)$. By Theorem 1.1.6, the irrationality exponent of a rational number is 1, while Dirichlet's theorem implies that $\mu(\alpha) \geq 2$ for every irrational number α. The existence of irrational numbers with $\mu(\alpha) = 2$ follows from the following theorem of Liouville [**Lio44**].

Theorem 1.1.8 (Liouville's Theorem). *Let α be an algebraic irrational number of degree d. Then there exists a constant $L > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{Lq^d}$$

for every rational number $\frac{p}{q}$.

Combining this theorem with Dirichlet's theorem, it follows that every quadratic irrational has irrationality exponent 2. Indeed, suppose that a quadratic irrational α satisfied $\mu(\alpha) > 2$. Then, for some $\varepsilon > 0$, there would be infinitely many reduced fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| < q^{-2-\varepsilon/2}$.

On the other hand, Liouville's theorem gives a constant $L > 0$ such that $\left|\alpha - \frac{p}{q}\right| > \frac{1}{Lq^2}$ for all $\frac{p}{q}$. For sufficiently large q we have $\frac{1}{Lq^2} > q^{-2-\varepsilon/2}$, a contradiction.

In this way, the elementary problem of approximating an irrational number by rational numbers developed into a theory that measures arithmetic properties of numbers through the irrationality exponent. Pursuing this topic further would take us away from the main theme of the present text, so we close this discussion by recalling Roth's theorem [**Rot55**].

Theorem 1.1.9 (Roth's Theorem). *If α is an algebraic irrational number, then its irrationality exponent is* 2*.*

Roth's theorem is a decisive strengthening of Liouville's theorem. Not only quadratic irrationals but all algebraic irrational numbers have irrationality exponent at most 2. Hence any irrational number whose irrationality exponent is larger than 2 must be transcendental; Roth's theorem therefore also gives a powerful sufficient condition for transcendence.

1.2. Minimum Problems for the Lagrange and Markov Constants. In view of Dirichlet's and Liouville's theorems, for any irrational number α there are infinitely many rational numbers $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$ up to the exponent $n = 2$, while the exponent cannot be uniformly increased beyond this. The next natural problem is therefore to fix the exponent at $n = 2$ and ask how large the constant L can be.

For an irrational number α, the supremum of the real numbers L for which there exist infinitely many rational numbers $\frac{p}{q}$ satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}$$

is called the *Lagrange constant* of α and is denoted by $\mathcal{L}(\alpha)$. The problem above asks for the smallest possible value of the Lagrange constant. Hurwitz gave the answer in 1891 [**Hur91**].

Theorem 1.1.10 (Hurwitz's Theorem). *For every irrational number α one has $\mathcal{L}(\alpha) \geq \sqrt{5}$, and for example $\mathcal{L}(\alpha) = \sqrt{5}$ when $\alpha = \frac{1+\sqrt{5}}{2}$.*

Hurwitz also stated that the next smallest Lagrange constant after $\sqrt{5}$ is $2\sqrt{2}$, and wrote that this fact follows from Markov's work. The proof of this point, however, is not given in [**Hur91**]. Markov's work here refers to his study of indefinite binary quadratic forms, beginning with his 1879 paper [**Mar79**]. For a binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$, Markov put $D = b^2 - 4ac$ and considered the quantity $\mathcal{M}(Q)$ defined by

$$\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(x, y)| = \frac{\sqrt{D}}{\mathcal{M}(Q)}.$$

This value is called the *Markov constant*. He reduced the problem of determining the smallest Markov constants to continued fraction theory. The smallest possible Markov constant is $\sqrt{5}$, attained by $Q(x, y) = x^2 - xy - y^2$, and the next smallest value is $2\sqrt{2}$, attained by $Q(x, y) = x^2 - 2xy - y^2$. Thus both the problem for Lagrange constants and the problem for Markov constants are solved by continued fractions, and Hurwitz recognized this analogy. This is where Diophantine approximation and the theory of binary quadratic forms meet.

Markov refined his work in the subsequent paper [**Mar80**] and reached the result now called *Markov's theorem*; historically, this was about ten years before Hurwitz noticed the relevance of Markov's work to Lagrange constants.

THEOREM 1.1.11 (Markov's Theorem). *Let M be the set of positive integers that occur in positive integer solutions of*

$$x^2 + y^2 + z^2 = 3xyz.$$

If $\mathcal{M}_0$ denotes the set of Markov constants less than 3*, then*

$$\mathcal{M}_0 = \left\{ \frac{\sqrt{9m^2 - 4}}{m} \;\middle|\; m \in M \right\}.$$

The elements of M are called *Markov numbers*. They should not be confused with Markov constants. The theorem says that Markov constants less than 3 are completely described by Markov numbers. For example, the Markov number 1 gives $\sqrt{5}$, and the Markov number 2 gives $2\sqrt{2}$. These values also occur as Lagrange constants via continued fraction theory, but it took some time before this relation was organized in a clear form.

1.3. Research on the Lagrange and Markov Spectra. After the minimum problems for the Lagrange and Markov constants were solved, attention turned to the problem of understanding the sets formed by all such constants. Let $\mathcal{L}$ be the set of all Lagrange constants and let $\mathcal{M}$ be the set of all Markov constants. They are called the *Lagrange spectrum* and the *Markov spectrum*, respectively.[1]

The fact that these sets can be described by continued fractions is now formulated as follows.

THEOREM 1.1.12. *For a bi-infinite sequence $\mathbf{a} = (a_n)_{n\in\mathbb{Z}} \in \mathbb{Z}_{\geq 1}^{\mathbb{Z}}$, put*

$$\ell_n(\mathbf{a}) := [a_n; a_{n+1}, a_{n+2}, \dots] + [0; a_{n-1}, a_{n-2}, \dots].$$

Then

$$\mathcal{L} = \left\{ \limsup_{n\to+\infty} \ell_n(\mathbf{a}) \;\middle|\; \mathbf{a} \in \mathbb{Z}_{\geq 1}^{\mathbb{Z}} \right\}, \quad \mathcal{M} = \left\{ \sup_{n\in\mathbb{Z}} \ell_n(\mathbf{a}) \;\middle|\; \mathbf{a} \in \mathbb{Z}_{\geq 1}^{\mathbb{Z}} \right\}.$$

This characterization appears without proof in Perron's paper [**Per21**]. For the second equality, it is natural to regard the essential content as already present in Markov's paper [**Mar79**]; nevertheless, Perron's formulation in the above form has become important. These equalities are therefore often called *Perron's identity*.

Perron's identity allows both the Lagrange spectrum and the Markov spectrum to be treated as the limit superior or the supremum of a function on bi-infinite continued fraction sequences. From this point of view, one also naturally obtains the inclusion $\mathcal{L} \subset \mathcal{M}$. Moreover, below 3 the

[1] Markov's contribution to the Lagrange spectrum is also substantial, and some authors may prefer the name Markov–Lagrange spectrum. The author is sympathetic to this view. In this text, however, we use the standard name in order to avoid a cumbersome terminology and confusion with the Markov spectrum.

two spectra coincide completely, and the Lagrange constants are described by Markov numbers just as in Markov's theorem.

THEOREM 1.1.13. *For Lagrange constants below* 3*,*

$$\mathcal{L} \cap (-\infty, 3) = \left\{ \frac{\sqrt{9m^2-4}}{m} \;\middle|\; m \in M \right\}.$$

Since Perron's formulation, much work has been done on the parts of $\mathcal{L}$ and $\mathcal{M}$ above 3. The following topics are somewhat outside the main line of this text, but they are important for understanding the global structure of the spectra.

First, as mentioned above, one has $\mathcal{L} \subset \mathcal{M}$, and Freiman proved that this inclusion is strict [**Fre68**].

THEOREM 1.1.14. $\mathcal{L} \subsetneq \mathcal{M}$*. In other words,* $\mathcal{M} \setminus \mathcal{L} \neq \emptyset$*.*

Another historically important result was proved by Hall in 1947 [**Hal47**].

THEOREM 1.1.15. *The interval* $[6, \infty)$ *is contained in* $\mathcal{L}$*. Consequently,* $[6, \infty)$ *is also contained in* $\mathcal{M}$*.*

This means that every sufficiently large real number belongs to the Lagrange spectrum, and hence also to the Markov spectrum. Such a half-line is called a *Hall ray*. Freiman later determined, in 1975, the smallest possible initial point of such a ray [**Fre75**].

THEOREM 1.1.16. *The largest half-line contained in* $\mathcal{L}$ *is* $[c_F, \infty)$*, where*

$$c_F = \frac{2221564096 + 283748\sqrt{462}}{491993569} \approx 4.5278295661\cdots.$$

In particular, $c_F \in \mathcal{L}$*.*

The number c_F is called the *Freiman constant*. It follows that on $[c_F, \infty)$ both $\mathcal{L}$ and $\mathcal{M}$ contain an entire real half-line, and any set-theoretic difference between $\mathcal{L}$ and $\mathcal{M}$ is contained in the interval $[3, c_F)$.

The structure of $\mathcal{L}$ and $\mathcal{M}$ in the remaining interval $[3, c_F)$ is still an active subject of research. For example, Moreira proved in 2018 the following result from the viewpoint of Hausdorff dimension [**Mor18**].

THEOREM 1.1.17. *For every* $t \in \mathbb{R}$*,*

$$\dim_H(\mathcal{L} \cap (-\infty, t)) = \dim_H(\mathcal{M} \cap (-\infty, t)).$$

If this common value is denoted by $d(t)$*, then* $d(t)$ *is nondecreasing and*

$$\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3.$$

Furthermore, in 2024 Erazo–Lima–Matheus–Moreira–Vieira proved the following [**ELM**$^{+}$**24**].

THEOREM 1.1.18. $\inf(\mathcal{M} \setminus \mathcal{L}) = 3$.

These results show that $\mathcal{L}$ and $\mathcal{M}$ have closely related fractal structures inside $[3, c_F)$, while their set-theoretic difference already appears immediately after 3. In this way, the study of the Lagrange and Markov spectra, although rooted in classical continued fraction theory, continues to develop today.

1.4. Markov Numbers and Reduced Fractions. The Markov numbers that describe the part of the Lagrange and Markov spectra below 3 have been studied in many contexts beyond their original motivation in Diophantine approximation. A starting point for this development was Frobenius's 1913 paper [**Fro13**]. In that paper, Frobenius related Markov numbers to reduced fractions. This correspondence shows that Markov numbers are deeply connected with rational numbers, lattice points, and line segments in the plane, and it still plays a fundamental role in modern work on Markov numbers.

To explain this relation, take a positive reduced fraction $t = \frac{p}{q} \geq 1$. Frobenius constructed an integer m_t from this fraction as follows.

(1) For each $i = 0, 1, \ldots, p+q$, let r_i be the remainder of iq upon division by $p+q$. For each $i \in \{1, 2, \ldots, p+q-1\}$, write one letter c if $r_i < r_{i+1}$ and one letter d if $r_i > r_{i+1}$. This gives a word s in the letters c, d.

(2) Replace each c in s by $1, 1$ and each d by $2, 2$, thereby obtaining an integer sequence S. Consider the continued fraction $[2; S, 2]$ and define m_t to be the numerator of its reduced fraction representation.

We have the following theorem.

THEOREM 1.1.19. *The integer m_t constructed above is a Markov number. Moreover, this construction defines a map $t \mapsto m_t$ from the set of reduced fractions at least 1 to the set of Markov numbers other than 1.*

There are two main points in this theorem.

The first is that it labels Markov numbers by reduced fractions. A Markov number is originally defined as a number appearing in a positive integer solution of the Markov equation, and from that definition alone it is difficult to determine its position in the Markov tree or in the fraction-label parametrization. Fraction labels supply coordinates on the set of Markov numbers and make individual Markov numbers much easier to handle. This idea is now standard, and the same viewpoint will be used in this text under the name "fraction labels."

The second important point is that the construction admits a natural geometric interpretation in the plane. Let ℓ be the line segment joining $(0,0)$ and $(p+q, q)$. The y-coordinate of the intersection of ℓ with the vertical line $x = i$ is $\frac{iq}{p+q}$, and its fractional part is precisely $\frac{r_i}{p+q}$. Thus, studying the increase or decrease of r_i amounts to recording how the line segment ℓ moves inside each vertical strip $i \le x \le i+1$. Since $\frac{q}{p+q} < 1$, the segment ℓ meets at most one horizontal lattice line in each vertical strip. Hence, if $r_i < r_{i+1}$, the segment meets no horizontal lattice line in that strip, while if $r_i > r_{i+1}$, it crosses exactly one horizontal lattice line. In other words, although the word s appears to be constructed by congruence calculations, it actually records how a line segment of rational slope crosses the integer lattice.

This viewpoint suggests that the number-theoretic objects arising as positive integer solutions of the Markov equation are directly connected with lattice segments in the plane and with word combinatorics. This connection is one of the sources of later developments.

Frobenius also formulated a simple but very important conjecture about Markov numbers.

CONJECTURE 1.1.20 (Frobenius's Uniqueness Conjecture). *For any Markov number c, a positive integer solution of the Markov equation whose largest component is c is uniquely determined up to permutation of its components.*

In terms of fraction labels, this can be understood as the question of whether distinct reduced fractions give distinct Markov numbers.

CONJECTURE 1.1.21. *The fraction-label map $t \mapsto m_t$ is injective.*

Although the statement is concise, it is a difficult problem about the internal structure of Markov numbers, and it remains open in full generality. Frobenius's work not only gave a way to describe Markov numbers by reduced fractions, but also introduced a central problem that has continued to be studied ever since.

1.5. Matrix Realizations of Markov Numbers and Their Hyperbolic-Geometric Interpretation. Around the 1950s, a point of view developed in which Markov numbers are realized through elements of the modular group. One turning point was Cohn's 1955 paper [**Coh55**]. Cohn observed that traces of matrices in the commutator subgroup $[SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$ of the modular group are closely related to the Markov equation. For any generators A, B of $[SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$ one obtains the relation

$$(\operatorname{tr}(A))^2 + (\operatorname{tr}(B))^2 + (\operatorname{tr}(AB))^2 = \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB).$$

This gives the equation $x^2 + y^2 + z^2 = xyz$, which contains the same information as the Markov equation $x^2 + y^2 + z^2 = 3xyz$ up to a scaling. Indeed, from a solution (a, b, c) of the Markov equation one obtains a solution $(3a, 3b, 3c)$ of the former equation, and conversely positive integer solutions of the former equation give positive integer solutions of the Markov equation after division by 3. Thus the Markov equation can also be viewed as a problem about traces of matrices.

The importance of Cohn's work is not merely that an equation resembling the Markov equation appears. To a matrix $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL(2, \mathbb{Z})$ he associated the binary quadratic form $Q(x, y) = cx^2 + (d - a)xy - by^2$, thereby providing a way to reinterpret the minimum problem for indefinite binary quadratic forms studied by Markov in matrix language. This made clear that the Markov numbers appearing in continued fractions and binary quadratic forms also arise naturally in the theory of discrete groups.

This line of thought led to a more hyperbolic-geometric interpretation in Cohn's 1971 paper [**Coh71**]. If the commutator subgroup acts on the upper half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$, the quotient is a once-punctured torus. Conjugacy classes of group elements correspond to free homotopy classes of closed curves on the torus; in particular, primitive hyperbolic elements correspond to simple closed geodesics. If A is the corresponding matrix and $\ell(A)$ is the length of the geodesic, then $|\operatorname{tr}(A)| = 2\cosh\left(\frac{\ell(A)}{2}\right)$. Thus the fact that traces occur as three times Markov numbers means that Markov numbers are directly connected with lengths of simple closed geodesics on the once-punctured torus.

This hyperbolic-geometric interpretation is closely related to the fraction labels discussed above. If one regards the once-punctured torus as an ordinary torus with a marked point and passes to the universal cover, one obtains the plane with a lattice of marked points. Primitive closed curves on the torus lift to straight lines of rational slope joining lattice points in this plane. Hence fraction labels are closely related to the description of closed curves on the torus in the universal cover. In this sense, the theory of Markov numbers connects objects that at first appear unrelated: reduced fractions, lattice segments, continued fractions, binary quadratic forms, matrices, and hyperbolic surfaces. This circle of ideas is closely related to Penner's decorated Teichmüller theory [**Pen87**] and to later reinterpretations through cluster algebra theory.

1.6. Cluster Algebra Theory and Generalized Markov Numbers. Cluster algebras were introduced by Fomin–Zelevinsky [**FZ02, FZ07**], and their connections with higher Teichmüller theory and related geometric structures were developed by Fock–Goncharov [**FG06, FG09**]. This theory has had a major influence on the theory of Markov numbers. A cluster algebra is generated from collections of elements called clusters, whose entries are called cluster variables, together with exchange operations called mutations. Cluster algebras are deeply connected with the combinatorics of surfaces: for cluster algebras associated with surfaces, arcs correspond to cluster variables, triangulations correspond to clusters, and flips of triangulations correspond to mutations. Through the metric quantities called λ-lengths in decorated Teichmüller space, cluster variables are realized as geometric quantities attached to arcs on a surface. Thus changes of triangulations of a surface can be read as transformations of variables in an algebra.

When the surface is the once-punctured torus, this framework is directly related to the classical theory of Markov numbers. In the Markov cluster algebra associated with this surface, suitable specializations of cluster variables give Markov numbers, and the three variables in one cluster give a solution of the Markov equation. Fraction labels, the combinatorics of lattice segments, Cohn matrices, and closed curves on the once-punctured torus are all organized under the common language of cluster algebras. This direction was broadened by the generalized cluster algebras introduced by Chekhov–Shapiro [**CS14**]. Generalized cluster algebras form a wider class of algebras containing ordinary cluster algebras, obtained by generalizing the exchange rules used in mutation. A natural question is then how much of the symmetry and good combinatorics of the classical Markov cluster algebra remains in this generalized setting.

The *generalized Markov numbers*, introduced by Gyoda–Matsushita [**GM23a**], arose from this question. Since classical Markov numbers are connected, through the Markov cluster algebra, with reduced fractions, lattice segments, and curves on the torus, it is natural to ask whether Chekhov–Shapiro's generalized cluster algebras contain a well-behaved class with properties analogous to those of the classical Markov cluster algebra. The resulting equation is the following extension of the classical Markov equation:

$$x^2 + y^2 + z^2 + k_1yz + k_2zx + k_3xy = (3 + k_1 + k_2 + k_3)xyz.$$

Here k_1, k_2, k_3 are nonnegative integers, and the integers appearing in positive integer solutions of this equation are called (k_1, k_2, k_3)-*generalized Markov numbers*. This equation is not only a formal deformation of the classical Markov equation. It appears naturally when one tries, inside generalized cluster algebras, to preserve the symmetries and mutation-based generation mechanism familiar from the classical theory. Thus generalized Markov numbers give a way to reinterpret the structure behind the classical theory in a wider setting. Subsequent work has reconstructed many aspects of the classical theory for generalized Markov numbers. In Gyoda–Maruyama [**GM23b**], generalized Cohn matrices were introduced. In Gyoda–Maruyama–Sato [**GMS24**], Frobenius-type combinatorics was extended to the theory of generalized Markov numbers. The relations among reduced fractions, words, continued fractions, Markov numbers, matrices, and curves in the classical theory thus also appear in the generalized setting. Furthermore, the author's paper [**Gyo25**] connected generalized Markov numbers back to the theory of the Lagrange and Markov spectra. More precisely, it contains the following result.

THEOREM 1.1.22. *Let m be a (k_1, k_2, k_3)-generalized Markov number, and suppose that it appears as the i-th component of a positive integer solution of the (k_1, k_2, k_3)-generalized Markov equation. We set*

$$\Delta(k_1, k_2, k_3, m, i) := ((3 + k_1 + k_2 + k_3)m - k_i)^2 - 4.$$

Then

$$\frac{\sqrt{\Delta(k_1, k_2, k_3, m, i)}}{m} \in \mathcal{L}.$$

In particular, if

$$\mathcal{M}_{k_1,k_2,k_3} := \left\{ \frac{\sqrt{\Delta(k_1, k_2, k_3, m, i)}}{m} \;\middle|\; \begin{array}{l} m \text{ is a } (k_1, k_2, k_3)\text{-generalized Markov number} \\ \text{appearing as the } i\text{-th component of a positive integer} \\ \text{solution of the } (k_1, k_2, k_3)\text{-generalized Markov equation} \end{array} \right\},$$

then $\mathcal{M}_{k_1,k_2,k_3} \subset \mathcal{L}$.

Taking $(k_1, k_2, k_3) = (0, 0, 0)$ recovers the direction of Markov's theorem asserting that the classical discrete Markov values below 3 occur as Lagrange constants. The irrational numbers realizing these values as Lagrange constants, and the binary quadratic forms realizing them as Markov constants, can also be given explicitly using simple closed curves on the once-punctured torus. The proof does not proceed by a direct generalization of the classical proof of Markov's theorem; rather, it uses a cluster-algebraic reinterpretation of the combinatorics of Markov numbers.

These facts indicate that generalized Markov numbers fit naturally into the arithmetic, geometric, and combinatorial structures already present in the classical theory. In this sense, they form a natural extension of classical Markov numbers.

2. Organization of the Text

The text is organized as follows. Part I develops the classical theory of the Lagrange and Markov spectra. Part II introduces generalized Markov numbers, uses them to construct generalized discrete Markov spectra, and relates the classical Markov theorem to this generalized framework.

Chapter 2 summarizes the theory of continued fractions needed later. After reviewing reduced fractions and finite regular continued fractions, it treats infinite regular continued fractions, convergents, continued-fraction matrices, and the decomposition of irrational numbers into orbits under the unimodular group. The final section recalls Lagrange's characterization of quadratic irrationals by periodic continued fractions. This prepares the connection, used in Chapters 3 and 4, between Lagrange and Markov constants and quadratic irrationals or binary quadratic forms.

Chapter 3 deals with the Lagrange spectrum. We first define the Lagrange constant and give basic examples, and then interpret it as a limit superior of quantities obtained from convergents. By introducing the representation in terms of bi-infinite sequences, we formulate it in a way that can be compared with the Markov constant in the next chapter. For quadratic irrationals, we reduce to reduced quadratic irrationals up to $GL(2, \mathbb{Z})$-equivalence and show that the Lagrange constant can be computed explicitly from the periodic part and the associated matrix. This establishes the method for computing values from periodic sequences used from Chapter 5 onward.

Chapter 4 turns to the Markov spectrum. We define the Markov constant for binary quadratic forms and organize representatives using canonical reduced binary quadratic forms and unimodular group orbits. We then express the Markov constant by bi-infinite sequences in a form parallel to the Lagrange spectrum. Finally, we show that the Markov constant of a binary quadratic form with rational coefficients coincides with the Lagrange constant of the corresponding quadratic irrational. This clarifies that quadratic irrationals, rational-coefficient binary quadratic forms, and periodic bi-infinite sequences give the same values. This will be the key point when the generalized theory is connected to spectra in the second half of the text.

Chapter 5 begins Part II and introduces generalized Markov equations and generalized Markov numbers. We first give the definition and basic properties of the (k_1, k_2, k_3)-generalized Markov equation, and then construct the generalized Markov tree corresponding to the classical Markov tree. Through the correspondence with the Farey tree, we assign fraction labels to generalized Markov numbers and thereby organize the numbers appearing at vertices by reduced fractions. At the end of the chapter we introduce characteristic numbers, which later serve as auxiliary quantities for describing the components of generalized Cohn matrices. Thus the role of this chapter is to carry over the classical picture of Markov numbers and fraction labels to the generalized setting and to prepare the data needed for later computations.

Chapter 6 introduces fence posets and generalized Markov distances in order to describe the generalized Markov numbers obtained in the previous chapter in combinatorial and geometric language. We first associate a fence poset with a finite integer sequence and show that the number of its order ideals is closely related to continued fractions and continued-fraction matrices. Next, we prove skein relations for pairs of overlapping fence posets, thereby controlling the counts of order ideals through product identities. Finally, we attach generalized Markov lengths to plane curves and generalized arcs, and define generalized Markov distance as a minimization of these lengths. In this way, the theory of generalized Markov numbers is understood not merely as operations on abstract integer sequences, but as a geometric theory involving intersections and resolutions of curves.

Chapter 7 introduces generalized Cohn matrices and translates the numerical and curve-theoretic information from the preceding chapters into the language of 2×2 matrices. We first define the generalized Cohn tree and show that the entries of generalized Cohn matrices can be described explicitly using generalized Markov numbers and characteristic numbers. We then prove relations among characteristic numbers. Finally, by introducing generalized strongly admissible sequences, we show that generalized Cohn matrices can be expressed as products of elementary matrices. In this chapter, the arithmetic data of Chapter 5 and the combinatorial-geometric data of Chapter 6 are unified through matrix representations. In particular, it becomes clear that Cohn matrices, which play a central role in the classical theory, retain an essential role in the generalized setting.

Chapter 8 defines the generalized discrete Markov spectrum and presents the main spectral results discussed in this text. We first define a family of discrete values constructed from generalized Markov numbers, and then show that these values are realized as Lagrange constants of quadratic irrationals and as Markov constants of binary quadratic forms with rational coefficients. By specializing the general theory to $(k_1, k_2, k_3) = (0, 0, 0)$, we explain how the classical Markov theorem is embedded in the framework developed here. We then consider irrational-slope limits of the generalized strongly admissible sequences obtained from rational slopes, and show that a bi-infinite sequence obtained from a line of irrational slope avoiding the points of the lifted triangulation gives the boundary value $3 + k_1 + k_2 + k_3$. We also discuss the correspondence between the $(0, 0, 0)$ type and the $(2, 2, 2)$ type, and then consider a generalization of Frobenius's uniqueness conjecture.

Chapter 9 collects several related directions for further reading and places the constructions of the text in a broader context.

Readers who are already familiar with continued fractions and the classical Lagrange and Markov spectra may start with Chapter 5, referring back to Part I as needed.

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Part 1

Lagrange and Markov Spectra

CHAPTER 2

Continued Fractions

The main theme of this text is the approximation of irrational numbers by rational numbers. Continued fractions that converge to a given irrational number are indispensable for studying such approximations. This chapter collects the basic facts about continued fractions that will be used from Chapter 3 onward.

Although we call them facts about continued fractions, a large part of the theory is, in effect, a theory of products of matrices in $GL(2,\mathbb{Z})$. Continued-fraction calculations can be interpreted as products of such matrices. For this reason, matrix calculations in $GL(2,\mathbb{Z})$ are an unavoidable tool in the modern treatment of continued fractions.

We first recall the elementary notions concerning reduced fractions. We then discuss finite regular continued-fraction expansions of rational numbers and infinite regular continued-fraction expansions of irrational numbers. In the final section we prove Lagrange's characterization of quadratic irrationals by periodic continued fractions.

The exposition and organization of this chapter are based largely on the corresponding chapters of [**Kid22**].

1. Reduced Fractions

We begin with the notion of a reduced fraction. Although this is familiar to many readers, we fix the precise convention used in this text.

Definition 2.1.1. Let $a, b \in \mathbb{Z}$ and assume that $(a, b) \neq (0, 0)$. If there exists $k \in \mathbb{Z}$ such that $b = ak$, then a is called a *divisor* of b. We write this as $a \mid b$.

This definition of divisibility also applies when a or b is zero or negative. For example, if $a \neq 0$, then $a \mid 0$ always holds, and hence every nonzero integer is a divisor of 0. Conversely, if $b \neq 0$, then $0 \mid b$ never holds, so 0 is not a divisor of any nonzero integer. Since the pair $(0, 0)$ is excluded from the definition above, we do not consider $0 \mid 0$ here.

Definition 2.1.2. Let $a_1, \dots, a_n$ be integers that are not all zero. The *greatest common divisor* $\gcd(a_1, \dots, a_n)$ is the positive integer d satisfying the following two conditions:

(1) $d \mid a_i$ for every $i = 1, \dots, n$.
(2) If an integer c satisfies $c \mid a_i$ for every $i = 1, \dots, n$, then $c \mid d$.

The greatest common divisor always exists.

Definition 2.1.3. If

$$\gcd(a_1, a_2, \dots, a_n) = 1,$$

then $a_1, a_2, \dots, a_n$ are said to be *relatively prime*.

With this definition, it also makes sense to ask whether a pair involving 0, or a pair involving both positive and negative numbers, is relatively prime.

Example 2.1.4. Let us check from the definition whether 0 is relatively prime to some small integers.

- The divisors of 1 are ± 1, whereas 0 is divisible by every nonzero integer. Hence the greatest common divisor of 0 and 1 is 1. Thus 0 and 1 are relatively prime.
- The divisors of 2 are $\pm 1, \pm 2$. Hence the greatest common divisor of 0 and 2 is 2. Thus 0 and 2 are not relatively prime.

- The divisors of 2 are $\pm 1, \pm 2$, and the divisors of -3 are $\pm 1, \pm 3$. Hence the greatest common divisor of 2 and -3 is 1. Thus 2 and -3 are relatively prime.

We now define fractions and reducedness.

DEFINITION 2.1.5. Let $a, b \in \mathbb{R}$ and assume that $(a, b) \neq (0, 0)$. The formal symbol $\frac{a}{b}$ is called a *fraction*. If a and b are integers, if they are relatively prime, and if $b \geq 0$, then the fraction $\frac{a}{b}$ is said to be *reduced*.

There are several points to note about this definition. First, the symbol $\frac{a}{b}$ is a *formal symbol*; it is not itself a rational number. If $b \neq 0$, then we may identify $\frac{a}{b}$ with the real number ab^{-1}, and hence with a rational number. On the other hand, as formal fractions we may also consider symbols such as $\frac{1}{0}$, which do not correspond to rational numbers.

In this text, when $b \neq 0$, we will often identify the fraction $\frac{a}{b}$ with the corresponding rational number, and conversely we will also use fractions as symbols representing rational numbers. Thus, when $b \neq 0$, the distinction between fractions and rational numbers will not be important. Notice also that, under our convention, the reduced fraction representing an integer n is $\frac{n}{1}$, and a fraction such as $\frac{1}{-2}$ is not reduced. Finally, $\frac{2}{0}$ is not reduced, whereas $\frac{1}{0}$ and $\frac{-1}{0}$ are reduced fractions.

2. Finite Regular Continued Fractions

In this section and the next one, we review the basic properties of continued fractions. We first define finite regular continued fractions, which correspond to rational numbers, and study their elementary properties.

DEFINITION 2.2.1. Let $(a_i)_{i=0}^n = (a_0, a_1, \ldots, a_n)$ be a finite sequence of real numbers. Define

$$[a_0; a_1, a_2, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

The numbers a_i are called the *partial quotients*. Suppose that $a_0 \in \mathbb{Z}$, that $a_k \in \mathbb{Z}_{\geq 1}$ for every $1 \leq k \leq n$, and that $a_n \neq 1$ whenever $n \neq 0$. A continued fraction satisfying these conditions is called a *finite regular continued fraction*.

The last condition removes the ambiguity $[a_0; \ldots, a_{n-1}, 1] = [a_0; \ldots, a_{n-1} + 1]$.

DEFINITION 2.2.2. Given a finite continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ and an index $0 \leq k \leq n$, we call $[a_0; a_1, a_2, \ldots, a_k]$ its k-th *convergent*.

The following proposition computes the convergents.

PROPOSITION 2.2.3. *Let $(a_0, a_1, \ldots, a_n)$ be a finite sequence of real numbers. Define two sequences by*

$$p_0 = a_0 \qquad p_1 = a_0 a_1 + 1 \qquad p_k = a_k p_{k-1} + p_{k-2}, \tag{2.2.1}$$

$$q_0 = 1 \qquad q_1 = a_1 \qquad q_k = a_k q_{k-1} + q_{k-2}. \tag{2.2.2}$$

Then, for every $0 \leq k \leq n$,

$$[a_0; a_1, \ldots, a_k] = \frac{p_k}{q_k}.$$

Moreover, for $2 \leq k \leq n$, the right-hand side may be written as

$$\frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}.$$

Proof. The identities $[a_0] = \frac{p_0}{q_0}$, $[a_0; a_1] = \frac{p_1}{q_1}$, and $[a_0; a_1, a_2] = \frac{p_2}{q_2}$ follow by direct calculation. Let $k \geq 3$, and assume that the assertion has been proved up to $k-1$. Then

$$[a_0; a_1, \dots, a_k] = [a_0; a_1, \dots, a_{k-1} + \tfrac{1}{a_k}] = \frac{(a_{k-1} + \frac{1}{a_k})p_{k-2} + p_{k-3}}{(a_{k-1} + \frac{1}{a_k})q_{k-2} + q_{k-3}}$$

$$= \frac{a_k(a_{k-1}p_{k-2} + p_{k-3}) + p_{k-2}}{a_k(a_{k-1}q_{k-2} + q_{k-3}) + q_{k-2}} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}} = \frac{p_k}{q_k}.$$

Thus the assertion also holds for k. □

Notice that this proposition does *not* assume that $[a_0; a_1, a_2, \dots, a_n]$ is a finite regular continued fraction. Until Lemma 2.2.5, regularity will not be assumed.

The convergents are conveniently computed using matrices.

Theorem 2.2.4. *Let $(a_0, a_1, \dots, a_n)$ be a finite sequence and let $0 \leq k \leq n$. Put $p_{-1} = 1$ and $q_{-1} = 0$, and define p_k, q_k by (2.2.1) and (2.2.2). Then*

$$\begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.2.3}$$

Proof. For $k = 0$, the identity

$$\begin{bmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$$

is immediate from the definition. Let $k \geq 1$, and assume that the theorem has been proved up to $k-1$. Then

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{bmatrix} \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_k p_{k-1} + p_{k-2} & p_{k-1} \\ a_k q_{k-1} + q_{k-2} & q_{k-1} \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}.$$

Thus the formula holds for k as well. □

The matrix $\left[\begin{smallmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{smallmatrix}\right]$ is the *continued-fraction matrix* of $[a_0; a_1, \dots, a_k]$.

The next lemma follows immediately from the matrix formula. In what follows, unless otherwise stated, p_k and q_k are used in the sense of Theorem 2.2.4.

Lemma 2.2.5. *For a finite sequence $(a_0, a_1, \dots, a_n)$ and every $0 \leq k \leq n$, one has*

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k+1}. \tag{2.2.4}$$

Proof. Take determinants on both sides of (2.2.3). □

We now derive several consequences for finite regular continued fractions.

Corollary 2.2.6. *Let $[a_0; a_1, \dots, a_n]$ satisfy the conditions for a finite regular continued fraction, except that we also allow $a_n = 1$. Then $\frac{p_k}{q_k}$ is a reduced fraction for every $0 \leq k \leq n$.*

Proof. The case $n = 0$ is clear, so assume $n \neq 0$. We have $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{Z}_{\geq 1}$ for $1 \leq k \leq n$. It is immediate from the recurrence that $q_k > 0$. Let d_k be the greatest common divisor of p_k and q_k, and write $p_k = d_k p'_k$ and $q_k = d_k q'_k$. Then $p_k q_{k-1} - q_k p_{k-1} = d_k(p'_k q_{k-1} - q'_k p_{k-1}) = (-1)^{k+1}$. Since $d_k \in \mathbb{Z}_{\geq 1}$ and $p'_k q_{k-1} - q'_k p_{k-1} \in \mathbb{Z}$, we must have $d_k = 1$. Hence $\frac{p_k}{q_k}$ is reduced. □

Corollary 2.2.7. *Let $[a_0; a_1, \dots, a_n]$ satisfy the conditions for a finite regular continued fraction, except that we also allow $a_n = 1$. Then the sequence $(q_1, \dots, q_n)$ is strictly increasing, and $q_k \geq k$ for every $1 \leq k \leq n$.*

Proof. We prove $q_k \geq k$ by induction. For $k = 1$ this is clear from the definition. Let $k \geq 2$ and assume $q_{k-1} \geq k - 1$. Since $a_k \geq 1$, we obtain $q_k = a_k q_{k-1} + q_{k-2} \geq q_{k-1} + 1 \geq k$. The same inequality also shows that the sequence is strictly increasing. □

Lemma 2.2.5 can be rewritten as follows.

COROLLARY 2.2.8. *Let $[a_0; a_1, \dots, a_n]$ satisfy the conditions for a finite regular continued fraction, except that we also allow $a_n = 1$. Then, for every $1 \le k \le n$,*

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k+1}}{q_k q_{k-1}}. \tag{2.2.5}$$

PROOF. Divide both sides of (2.2.4) by $q_k q_{k-1}$. □

COROLLARY 2.2.9. *Let $[a_0; a_1, \dots, a_n]$ satisfy the conditions for a finite regular continued fraction, except that we also allow $a_n = 1$. Then, for all $k \in \mathbb{Z}_{\ge 0}$ and $\ell \in \mathbb{Z}_{\ge 1}$ satisfying $1 \le 2k + 2\ell + 1 \le n$, one has*

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+2\ell}}{q_{2k+2\ell}} < \frac{p_{2k+2\ell+1}}{q_{2k+2\ell+1}} < \frac{p_{2k+1}}{q_{2k+1}}.$$

PROOF. We first prove

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+2\ell}}{q_{2k+2\ell}} \quad \text{and} \quad \frac{p_{2k+2\ell+1}}{q_{2k+2\ell+1}} < \frac{p_{2k+1}}{q_{2k+1}}.$$

For $2 \le m \le n$, we compute

$$\begin{aligned}
\frac{p_m}{q_m} - \frac{p_{m-2}}{q_{m-2}} &= \frac{p_m q_{m-2} - p_{m-2} q_m}{q_m q_{m-2}} \\
&= \frac{(a_m p_{m-1} + p_{m-2}) q_{m-2} - p_{m-2}(a_m q_{m-1} + q_{m-2})}{q_m q_{m-2}} \\
&= \frac{a_m (p_{m-1} q_{m-2} - q_{m-1} p_{m-2})}{q_m q_{m-2}} = \frac{(-1)^m a_m}{q_m q_{m-2}},
\end{aligned}$$

where the last equality follows from (2.2.4). Since $a_m > 0$, taking $m = 2k + 2$ gives $\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+2}}{q_{2k+2}}$, and taking $m = 2k + 3$ gives $\frac{p_{2k+3}}{q_{2k+3}} < \frac{p_{2k+1}}{q_{2k+1}}$. These are the desired inequalities for $\ell = 1$. The inequalities for general ℓ follow by repeated application and transitivity.

It remains to prove

$$\frac{p_{2k+2\ell}}{q_{2k+2\ell}} < \frac{p_{2k+2\ell+1}}{q_{2k+2\ell+1}}.$$

Substituting $2k + 2\ell + 1$ for k in (2.2.5), we obtain

$$\frac{p_{2k+2\ell+1}}{q_{2k+2\ell+1}} - \frac{p_{2k+2\ell}}{q_{2k+2\ell}} = \frac{(-1)^{2k+2\ell+2}}{q_{2k+2\ell+1} q_{2k+2\ell}} > 0.$$

This proves the assertion. □

The indexing in Corollary 2.2.9 may obscure the simple meaning of the statement. Applying it for all possible k gives, for example,

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1},$$

which may be easier to visualize.

We finish this section by proving the bijective correspondence between rational numbers and finite regular continued fractions. We first define the set of sequences that represent finite regular continued fractions. Since finite sequences have varying lengths, we regard them as infinite sequences that become zero from some point on. Define

$$\mathscr{Z} := \{(a_k)_{k=0}^{\infty} \mid a_0 \in \mathbb{Z},\ a_i = 0 \text{ for every } i \in \mathbb{Z}_{\ge 1}\}.$$

The map $z\colon \mathscr{Z} \to \mathbb{Z}$ defined by $z((a_k)_{k=0}^{\infty}) = a_0$ is clearly a bijection. Thus z gives a bijective correspondence between integers and their finite regular continued-fraction expansions.

Next consider rational numbers that are not integers. Define

$$\mathscr{Q} := \left\{ (a_k)_{k=0}^{\infty} \;\middle|\; \begin{array}{l} a_0 \in \mathbb{Z},\ \exists n \in \mathbb{Z}_{\ge 1} \text{ such that } a_1, \dots, a_{n-1} \in \mathbb{Z}_{\ge 1}, \\ a_n \in \mathbb{Z}_{\ge 2},\ \text{and } a_i = 0 \text{ for all } i \in \mathbb{Z}_{\ge n+1} \end{array} \right\}.$$

Define $f\colon \mathscr{Q} \to \mathbb{Q} \setminus \mathbb{Z}$ by

$$f((a_k)_{k=0}^{\infty}) = [a_0; a_1, \dots, a_n],$$

where n is the integer such that $a_n \in \mathbb{Z}_{\geq 2}$ and $a_i = 0$ for all $i \geq n+1$. It is not immediate that this map is bijective, so we construct its inverse. We begin with the following lemma.

LEMMA 2.2.10. *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$. Construct α_k and a_k recursively by*

$$\alpha_0 = \alpha, \qquad a_k = \lfloor \alpha_k \rfloor, \qquad \alpha_{k+1} = \frac{1}{\alpha_k - a_k}. \tag{2.2.6}$$

Then there always exists $n \in \mathbb{Z}_{>0}$ such that $a_n = \alpha_n$. Thus the process stops at that point.

PROOF. Write $\alpha = \alpha_0 = \frac{r_0}{s_0}$ as a reduced fraction. Since $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, we have $s_0 \geq 2$. As long as $a_{k-1} \neq \alpha_{k-1}$, the number α_k is rational; write it as a reduced fraction $\alpha_k = \frac{r_k}{s_k}$. If $s_k = 1$, then $\alpha_k \in \mathbb{Z}$, hence $a_k = \alpha_k$, and we may take $n = k$.

Assume $s_k \geq 2$. Then

$$\alpha_k - a_k = \frac{r_k}{s_k} - a_k = \frac{r_k - a_k s_k}{s_k}.$$

Since $\alpha_k - a_k \neq 0$ and $0 < \alpha_k - a_k < 1$, we have $0 < r_k - a_k s_k < s_k$. By definition,

$$\alpha_{k+1} = \frac{s_k}{r_k - a_k s_k}.$$

If this is written as the reduced fraction $\frac{r_{k+1}}{s_{k+1}}$, then s_{k+1} divides $r_k - a_k s_k$. Hence

$$s_{k+1} \leq r_k - a_k s_k < s_k.$$

Thus, as long as $s_k \geq 2$, the denominators strictly decrease. Since $s_k \geq 1$, there must be some n such that $s_n = 1$. Then $\alpha_n \in \mathbb{Z}$, so $a_n = \alpha_n$. □

REMARK 2.2.11. The procedure in Lemma 2.2.10 is exactly the Euclidean algorithm. In the next section we will carry out the analogous procedure for infinite continued fractions.

THEOREM 2.2.12. *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$. Use (2.2.6) and let n be the smallest index such that $a_n = \alpha_n$. Consider the finite sequence $(a_k)_{k=0}^n$, and extend it by putting $a_i = 0$ for all $i \geq n+1$. Then the resulting infinite sequence $(a_k)_{k=0}^\infty$ belongs to $\mathscr{Q}$.*

PROOF. For every $k \geq 0$, the equality $a_k = \lfloor \alpha_k \rfloor$ gives $a_k \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Z}$, we have $n \neq 0$. It remains to prove that $a_1, \ldots, a_{n-1} \geq 1$ and $a_n \geq 2$. For every $0 \leq k \leq n-1$, we have $\alpha_k - a_k \neq 0$, hence $0 < \alpha_k - a_k < 1$. Therefore

$$\alpha_{k+1} = \frac{1}{\alpha_k - a_k} > 1,$$

and so $a_{k+1} \geq 1$. Thus $a_1, \ldots, a_n \geq 1$.

It remains to show $a_n \geq 2$. Suppose $a_n = 1$. Since $a_n = \alpha_n$, we would have $\alpha_n = 1$. Then (2.2.6) gives

$$\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n} = a_{n-1} + 1,$$

so $\alpha_{n-1} \in \mathbb{Z}$. Hence $\alpha_{n-1} = a_{n-1}$, contradicting the minimality of n. Therefore $a_n \geq 2$. □

Theorem 2.2.12 says that the correspondence $g\colon \mathbb{Q} \setminus \mathbb{Z} \to \mathscr{Q}$ defined by $g(\alpha) = (a_k)_{k=0}^\infty$ is well-defined. We now prove that f and g are inverse maps.

THEOREM 2.2.13. *The maps f and g are inverse to each other. In particular, f is a bijection.*

Before proving this theorem, we record a lemma.

LEMMA 2.2.14. *For a finite regular continued fraction $[a_0; a_1, \ldots, a_n]$, one has*

$$a_0 = \lfloor [a_0; a_1, \ldots, a_n] \rfloor.$$

PROOF. If $n = 0$, then the assertion is clear. Assume $n \geq 1$. It suffices to prove

$$a_0 \leq [a_0; a_1, \ldots, a_n] < a_0 + 1,$$

or equivalently

$$0 < [0; a_1, \ldots, a_n] < 1.$$

If $n = 1$, then the last partial quotient satisfies $a_1 \geq 2$, and hence

$$[0; a_1] = \frac{1}{a_1} < 1.$$

If $n \geq 2$, then

$$[0; a_1, \dots, a_n] = \frac{1}{a_1 + \frac{1}{[a_2; \dots, a_n]}} < \frac{1}{a_1} \leq 1.$$

Positivity is clear, and the claim follows. □

Proof of Theorem 2.2.13. We first show $f \circ g = \mathrm{id}_{\mathbb{Q} \setminus \mathbb{Z}}$. It suffices to prove the following: for $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, if $(\alpha_k)_{k=0}^n$ and $(a_k)_{k=0}^n$ are constructed by (2.2.6), then for every $0 \leq k \leq n-1$,

$$\alpha = [a_0; a_1, \dots, a_k, \alpha_{k+1}] = [a_0; a_1, \dots, a_n].$$

We prove the first equality. For $k = 0$ it follows by direct computation. Suppose $k \geq 1$ and assume

$$\alpha = [a_0; a_1, \dots, a_{k-1}, \alpha_k].$$

Then

$$[a_0; a_1, \dots, a_{k-1}, a_k, \alpha_{k+1}] = [a_0; a_1, \dots, a_{k-1}, a_k + \alpha_k - a_k] = [a_0; a_1, \dots, a_{k-1}, \alpha_k] = \alpha.$$

Taking $k = n-1$ and using $a_n = \alpha_n$ gives the second equality. Hence $f \circ g = \mathrm{id}_{\mathbb{Q} \setminus \mathbb{Z}}$.

Next we show $g \circ f = \mathrm{id}_{\mathscr{Q}}$. Take $(b_k)_{k=0}^\infty \in \mathscr{Q}$, and write

$$f((b_k)_{k=0}^\infty) = [b_0; b_1, \dots, b_n].$$

Let $(a_k)_{k=0}^\infty$ be the sequence constructed from this rational number by (2.2.6), with undefined terms completed by zero after the process stops. We must show that $(a_k)_{k=0}^\infty = (b_k)_{k=0}^\infty$. By Lemma 2.2.14, we first obtain $a_0 = b_0$. Hence

$$\alpha_1 = \frac{1}{[b_0; b_1, \dots, b_n] - b_0} = \frac{1}{[0; b_1, \dots, b_n]} = [b_1; b_2, \dots, b_n].$$

Applying Lemma 2.2.14 again gives $a_1 = b_1$. Repeating the same argument shows that $(a_k)_{k=0}^n = (b_k)_{k=0}^n$. Finally, since

$$\alpha_n = [b_n] = b_n = a_n,$$

the algorithm stops at this point, and hence $a_i = 0$ for all $i \geq n+1$. Thus $(a_k)_{k=0}^\infty = (b_k)_{k=0}^\infty$. □

Since the sequences in $\mathscr{Z} \cup \mathscr{Q}$ correspond to finite regular continued fractions, and since $z\colon \mathscr{Z} \to \mathbb{Z}$ and $f\colon \mathscr{Q} \to \mathbb{Q} \setminus \mathbb{Z}$ are bijections, we obtain the following theorem.

Theorem 2.2.15. *For every rational number α, there exists a unique finite regular continued fraction whose value is α. The sequence of partial quotients is obtained by the algorithm* (2.2.6).

Although Lemma 2.2.10 was stated for nonintegral rational numbers, the same procedure includes the integral case as a special case.

Definition 2.2.16. The finite regular continued fraction whose value is a rational number α is called the *finite regular continued-fraction expansion* of α.

3. Infinite Continued Fraction Expansions of Irrational Numbers

In the preceding section, we saw that finite regular continued fractions are in bijection with rational numbers. By extending the integer sequences that define finite regular continued fractions to infinite sequences with no zero terms after the first entry, and by taking the corresponding limit, one obtains continued fractions representing irrational numbers. We justify this construction in this section. The strategy is almost the same as in the rational case, but limits make some parts slightly more delicate.

Consider the set

$$\mathscr{S} := \{(a_k)_{k=0}^\infty \mid a_0 \in \mathbb{Z},\ a_k \in \mathbb{Z}_{\geq 1}\ (k \geq 1)\}.$$

We define the infinite analogue of finite regular continued fractions as follows.

DEFINITION 2.3.1. For $(a_k)_{k=0}^{\infty} \in \mathscr{S}$, consider

$$\lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n].$$

This limit, when it exists, is called an *infinite regular continued fraction.*

We also write an infinite regular continued fraction as

$$[a_0; a_1, a_2, \ldots], \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

in order to display the underlying infinite sequence of partial quotients.

Since the definition involves a limit, it is not a priori clear that an infinite regular continued fraction has a real value. The next theorem shows that it does, and that the value is irrational.

THEOREM 2.3.2. *For every $(a_k)_{k=0}^{\infty} \in \mathscr{S}$, the limit*

$$\lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n]$$

exists and is irrational.

PROOF. Put

$$b_k := [a_0; a_1, \ldots, a_k] = \frac{p_k}{q_k}.$$

Extending Corollary 2.2.9 to the infinite sequence, we see that the sequence $\{b_{2k+1}\}_{k=0}^{\infty}$ is decreasing and bounded, and that $\{b_{2k}\}_{k=0}^{\infty}$ is increasing and bounded. Hence the limits

$$\lim_{k\to\infty} b_{2k+1} \quad \text{and} \quad \lim_{k\to\infty} b_{2k}$$

exist as real numbers. Denote them by α_1 and α_2, respectively. We prove that $\alpha_1 = \alpha_2$. By Corollaries 2.2.8 and 2.2.7, for $k \geq 1$ we have

$$0 < b_{2k+1} - b_{2k} = \frac{(-1)^{2k+2}}{q_{2k+1}q_{2k}} \leq \frac{1}{2k(2k+1)}.$$

Letting $k \to \infty$, we obtain $\lim_{k\to\infty}(b_{2k+1} - b_{2k}) = 0$. Hence

$$\alpha_1 - \alpha_2 = \lim_{k\to\infty} b_{2k+1} - \lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} (b_{2k+1} - b_{2k}) = 0.$$

Thus the two limits coincide. Let $\alpha := \alpha_1 = \alpha_2$.

We next prove that α is irrational. For every $k \geq 0$ we have

$$b_{2k} < \alpha < b_{2k+1}.$$

Therefore

$$0 < \alpha - b_{2k} < b_{2k+1} - b_{2k} < \frac{1}{q_{2k}q_{2k+1}}.$$

Since $b_{2k} = \frac{p_{2k}}{q_{2k}}$, multiplying by q_{2k} gives

$$0 < \alpha q_{2k} - p_{2k} < \frac{1}{q_{2k+1}}.$$

Suppose, for a contradiction, that $\alpha = \frac{a}{b}$ is rational, where $\frac{a}{b}$ is reduced and $b > 0$. Then

$$0 < a q_{2k} - b p_{2k} < \frac{b}{q_{2k+1}}.$$

The middle term is an integer for every k, while $\frac{b}{q_{2k+1}} < 1$ for all sufficiently large k by Corollary 2.2.7. This is impossible. Hence α is irrational. □

The theorem implies that the correspondence

$$F\colon \mathscr{S} \to \mathbb{R} \setminus \mathbb{Q}, \qquad F((a_k)_{k=0}^{\infty}) = \lim_{n\to\infty} [a_0; a_1, \ldots, a_n],$$

is well-defined. We now construct its inverse. First we prove the following theorem.

THEOREM 2.3.3. *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. *Construct* $(\alpha_k)_{k=0}^{\infty}$ *and* $(a_k)_{k=0}^{\infty}$ *by*

$$\alpha_0 = \alpha, \qquad a_k = \lfloor \alpha_k \rfloor, \qquad \alpha_{k+1} = \frac{1}{\alpha_k - a_k}. \tag{2.3.1}$$

Then $(a_k)_{k=0}^{\infty} \in \mathscr{S}$.

PROOF. For every $k \geq 0$, $a_k = \lfloor \alpha_k \rfloor$, so $a_k \in \mathbb{Z}$. Moreover, every α_k is irrational. Indeed, if some α_k were rational, then

$$\alpha_{k-1} = a_{k-1} + \frac{1}{\alpha_k}$$

would also be rational, and repeating this argument would imply that α_0 is rational, a contradiction. Hence, for every $k \geq 1$, we have

$$0 < \alpha_{k-1} - a_{k-1} < 1,$$

and therefore $\alpha_k > 1$. It follows that $a_k \geq 1$ for every $k \geq 1$. Thus $(a_k)_{k=0}^{\infty} \in \mathscr{S}$. □

Let $G\colon \mathbb{R} \setminus \mathbb{Q} \to \mathscr{S}$ be the correspondence in this theorem; that is, $G(\alpha) = (a_n)_{n=0}^{\infty}$, where the sequence is constructed by (2.3.1). Theorem 2.3.3 says that G is well-defined. The main result of this section is the following.

THEOREM 2.3.4. *The maps* F *and* G *are inverse to each other. In particular,* F *is a bijection.*

Before proving this theorem, we record a lemma.

LEMMA 2.3.5. *For an infinite regular continued fraction, one has*

$$a_0 = \left\lfloor \lim_{n\to\infty} [a_0; a_1, \ldots, a_n] \right\rfloor.$$

PROOF. It suffices to prove

$$a_0 < \lim_{n\to\infty} [a_0; a_1, \ldots, a_n] < a_0 + 1,$$

or equivalently

$$0 < \lim_{n\to\infty} [0; a_1, \ldots, a_n] < 1.$$

Apply the ordering of even and odd convergents, proved in Theorem 2.3.2, to the infinite continued fraction $[0; a_1, a_2, \ldots]$. Its value lies strictly between its first two convergents. Hence

$$0 < \lim_{n\to\infty} [0; a_1, \ldots, a_n] < \frac{1}{a_1} \leq 1,$$

and the desired inequality follows. □

PROOF OF THEOREM 2.3.4. We first show $F \circ G = \mathrm{id}_{\mathbb{R}\setminus\mathbb{Q}}$. It suffices to prove the following: for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, construct $(\alpha_k)_{k=0}^{\infty}$ and $(a_k)_{k=0}^{\infty}$ by (2.3.1). Then, for every $k \in \mathbb{Z}_{\geq 0}$,

$$\alpha = [a_0; a_1, \ldots, a_k, \alpha_{k+1}] = \lim_{n\to\infty} [a_0; a_1, \ldots, a_n].$$

We first prove the first equality. For $k = 0$ it follows by direct computation. Suppose $k \geq 1$ and assume

$$\alpha = [a_0; a_1, \ldots, a_{k-1}, \alpha_k].$$

Then

$$[a_0; a_1, \ldots, a_{k-1}, a_k, \alpha_{k+1}] = [a_0; a_1, \ldots, a_{k-1}, a_k + \alpha_k - a_k] = [a_0; a_1, \ldots, a_{k-1}, \alpha_k] = \alpha.$$

We now prove

$$\alpha = \lim_{n\to\infty} [a_0; a_1, \ldots, a_n].$$

By the equality just proved and Proposition 2.2.3, for every $n \geq 1$ we have

$$\alpha = [a_0; a_1, \ldots, a_n, \alpha_{n+1}] = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}.$$

Therefore

$$\alpha - \frac{p_n}{q_n} = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n}$$

$$= -\frac{p_n q_{n-1} - q_n p_{n-1}}{(\alpha_{n+1} q_n + q_{n-1}) q_n} = -\frac{(-1)^{n+1}}{(\alpha_{n+1} q_n + q_{n-1}) q_n}, \tag{2.3.2}$$

where the last equality follows from (2.2.4). Since $\alpha_{n+1} > a_{n+1}$, we have

$$\alpha_{n+1} q_n + q_{n-1} > a_{n+1} q_n + q_{n-1} = q_{n+1} > n + 1,$$

where the last inequality follows from Corollary 2.2.7. Hence

$$\left| \alpha - \frac{p_n}{q_n} \right| = \frac{1}{(\alpha_{n+1} q_n + q_{n-1}) q_n} < \frac{1}{q_{n+1} q_n} < \frac{1}{n(n+1)}. \tag{2.3.3}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \left| \alpha - \frac{p_n}{q_n} \right| = 0.$$

Since the limit $\lim_{n\to\infty} p_n/q_n$ exists by Theorem 2.3.2, this estimate implies

$$\alpha = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} [a_0; a_1, \ldots, a_n].$$

This proves $F \circ G = \mathrm{id}_{\mathbb{R}\setminus\mathbb{Q}}$.

Next we show $G \circ F = \mathrm{id}_{\mathscr{S}}$. Take $(b_n)_{n=0}^{\infty} \in \mathscr{S}$, and put

$$\alpha = \lim_{n\to\infty} [b_0; b_1, \ldots, b_n].$$

Let $(a_n)_{n=0}^{\infty}$ be the sequence obtained from α by (2.3.1). We must prove $(a_n)_{n=0}^{\infty} = (b_n)_{n=0}^{\infty}$. By Lemma 2.3.5, we first get $a_0 = b_0$. Hence

$$\begin{aligned} \alpha_1 &= \frac{1}{\lim_{n\to\infty}[b_0; b_1, \ldots, b_n] - b_0} = \lim_{n\to\infty} \frac{1}{[b_0; b_1, \ldots, b_n] - b_0} \\ &= \lim_{n\to\infty} \frac{1}{[0; b_1, \ldots, b_n]} = \lim_{n\to\infty} [b_1; b_2, \ldots, b_n]. \end{aligned}$$

Applying Lemma 2.3.5 again gives $a_1 = b_1$. Repeating the same argument proves $(a_n)_{n=0}^{\infty} = (b_n)_{n=0}^{\infty}$. □

This bijection yields the following statement.

THEOREM 2.3.6. *For every irrational number α, there exists a unique infinite regular continued fraction whose limit is α. The sequence of partial quotients is obtained by the algorithm (2.3.1).*

DEFINITION 2.3.7. The infinite regular continued fraction whose limit is an irrational number α is called the *infinite regular continued-fraction expansion* of α.

It is natural to expect that truncating the infinite continued-fraction expansion of an irrational number α gives good rational approximations to α. Conversely, it is also known that all sufficiently good rational approximations arise as such truncations. We now explain this. First we record an inequality that follows from the preceding discussion.

PROPOSITION 2.3.8. *Let α be irrational, and let $\frac{p_n}{q_n}$ be its n-th convergent. Then*

$$\frac{1}{q_n q_{n+2}} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

PROOF. The second inequality follows from (2.3.3). For the first one, use (2.3.2). Since $\alpha_{n+1} = a_{n+1} + 1/\alpha_{n+2} < a_{n+1} + 1$, we estimate the denominator as follows:

$$(\alpha_{n+1} q_n + q_{n-1}) q_n < ((a_{n+1} + 1) q_n + q_{n-1}) q_n = (q_{n+1} + q_n) q_n \le (a_{n+2} q_{n+1} + q_n) q_n = q_{n+2} q_n.$$

This gives the desired lower bound. □

The next proposition says that convergents are especially good approximations.

PROPOSITION 2.3.9. *Let α be irrational and let $n \geq 1$. For the n-th convergent $\frac{p_n}{q_n}$ of α, and for any rational number $\frac{p}{q}$ satisfying*

$$\frac{p_n}{q_n} \neq \frac{p}{q}, \qquad 0 < q \leq q_n,$$

one has

$$|q\alpha - p| \geq |q_{n-1}\alpha - p_{n-1}| > |q_n\alpha - p_n|.$$

In particular,

$$\left|\alpha - \frac{p}{q}\right| > \left|\alpha - \frac{p_n}{q_n}\right|.$$

PROOF. If $\frac{p}{q}$ is not reduced, reducing it preserves the assumptions of the proposition. Thus we may assume from the beginning that $\frac{p}{q}$ is reduced. By the lower bound in Proposition 2.3.8, we have

$$\frac{1}{q_{n+1}} < |q_{n-1}\alpha - p_{n-1}|,$$

and by the upper bound we have

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}}.$$

Thus

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}} < |q_{n-1}\alpha - p_{n-1}|,$$

which gives the second inequality in the first assertion.

It remains to prove the first inequality. Consider the linear system

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Since $\left[\begin{smallmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{smallmatrix}\right] \in GL(2,\mathbb{Z})$, the system has an integral solution. Solving it gives

$$c = (-1)^{n+1}(pq_{n-1} - p_{n-1}q), \qquad d = (-1)^{n+1}(p_nq - q_np).$$

Here $c, d \in \mathbb{Z}$, and the assumption $\frac{p}{q} \neq \frac{p_n}{q_n}$ implies $d \neq 0$.

If $c = 0$, then the system gives $dp_{n-1} = p$ and $dq_{n-1} = q$, hence

$$|q\alpha - p| = |d|\,|q_{n-1}\alpha - p_{n-1}| \geq |q_{n-1}\alpha - p_{n-1}|,$$

as desired.

Now assume $c \neq 0$. Since $q \leq q_n$, if c and d had the same sign, then they would both have to be positive, since $q = cq_n + dq_{n-1} > 0$. Hence $q = cq_n + dq_{n-1}$ would imply $q \geq q_n$, with equality only when $c = 1$ and $d = 0$. In that case we would have $p = p_n$ and $q = q_n$, contradicting the assumption. Hence c and d have opposite signs.

On the other hand, the proof of Theorem 2.3.2 shows that either

$$\frac{p_{n-1}}{q_{n-1}} < \alpha < \frac{p_n}{q_n} \quad \text{or} \quad \frac{p_n}{q_n} < \alpha < \frac{p_{n-1}}{q_{n-1}}.$$

Thus $q_{n-1}\alpha - p_{n-1}$ and $q_n\alpha - p_n$ have opposite signs. It follows that

$$c(q_n\alpha - p_n) \quad \text{and} \quad d(q_{n-1}\alpha - p_{n-1})$$

have the same sign. Therefore

$$\begin{aligned} |q\alpha - p| &= |(cq_n + dq_{n-1})\alpha - (cp_n + dp_{n-1})| \\ &= |c(q_n\alpha - p_n) + d(q_{n-1}\alpha - p_{n-1})| \\ &= |c(q_n\alpha - p_n)| + |d(q_{n-1}\alpha - p_{n-1})| \\ &\geq |d(q_{n-1}\alpha - p_{n-1})| \geq |q_{n-1}\alpha - p_{n-1}|. \end{aligned}$$

This proves the first assertion.

The final assertion follows from

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q}|q\alpha - p| > \frac{1}{q}|q_n\alpha - p_n| \geq \frac{1}{q_n}|q_n\alpha - p_n| = \left|\alpha - \frac{p_n}{q_n}\right|.$$

□

Using this, we prove that every sufficiently good approximation comes from a convergent.

THEOREM 2.3.10. *Let α be irrational. If a reduced fraction $\frac{p}{q}$ with $q > 0$ satisfies*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent of α; equivalently, it is obtained by truncating the continued-fraction expansion of α.

PROOF. Suppose, for a contradiction, that $\frac{p}{q} \neq \frac{p_k}{q_k}$ for every convergent $\frac{p_k}{q_k}$ of α. Since α is irrational, Corollary 2.2.7 implies that there exists N such that $q < q_N$. Choose the smallest n such that $q_n > q$. Then

$$q_{n-1} \leq q < q_n.$$

By Proposition 2.3.9 and the assumption,

$$|q_{n-1}\alpha - p_{n-1}| \leq |q\alpha - p| < \frac{1}{2q}.$$

Therefore

$$\frac{1}{qq_{n-1}} \leq \frac{|qp_{n-1} - pq_{n-1}|}{qq_{n-1}} = \left|\frac{p_{n-1}}{q_{n-1}} - \frac{p}{q}\right| \leq \left|\frac{p_{n-1}}{q_{n-1}} - \alpha\right| + \left|\alpha - \frac{p}{q}\right| < \frac{1}{2qq_{n-1}} + \frac{1}{2q^2}.$$

This inequality implies $q < q_{n-1}$, contradicting $q_{n-1} \leq q$. Hence $\frac{p}{q}$ is a convergent of α. □

REMARK 2.3.11. Proposition 2.3.8, Proposition 2.3.9, and Theorem 2.3.10 also have rational analogues. If α is rational and its finite regular continued-fraction expansion has length $N + 1$, the same arguments work by restricting n to $0 \leq n \leq N - 1$. In the proof of Theorem 2.3.10, the step in which one chooses N with $q < q_N$ should be replaced by the following argument: for $\alpha = \frac{p_N}{q_N}$,

$$\frac{1}{q_N} \leq \frac{|qp_N - pq_N|}{q_N} = |q\alpha - p| < \frac{1}{2q},$$

which implies $q < q_N$.

4. Unimodular Group Orbits of Irrational Numbers

In this section we decompose irrational numbers into orbits under a matrix group. The reason this decomposition is useful will become clear in the next chapter: irrational numbers lying in the same orbit have the same Lagrange constant. Hence, when studying possible values of Lagrange constants, this orbit decomposition removes redundant work. We first introduce the group that acts on irrational numbers.

DEFINITION 2.4.1. Consider the set

$$GL(2, \mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z},\ |ad - bc| = 1 \right\}.$$

It is a group under matrix multiplication. We call it the *unimodular group*.

For $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathbb{Z})$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, define the action of A on α by the fractional linear transformation

$$A\alpha := \frac{a\alpha + b}{c\alpha + d}.$$

We have the following.

PROPOSITION 2.4.2. *For $A \in GL(2, \mathbb{Z})$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, one has $A\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, the fractional linear transformations of $GL(2, \mathbb{Z})$ define a left action*

$$GL(2, \mathbb{Z}) \curvearrowright \mathbb{R} \setminus \mathbb{Q}.$$

PROOF. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Suppose $A\alpha$ is rational. Then

$$\alpha = A^{-1}(A\alpha) = \frac{d(A\alpha) - b}{-c(A\alpha) + a}$$

would also be rational, a contradiction. Hence $A\alpha$ is irrational.

Let $E_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then $E_2\alpha = \alpha$. If $B = \left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right]$, then

$$A(B\alpha) = A\frac{e\alpha + f}{g\alpha + h} = \frac{a\frac{e\alpha+f}{g\alpha+h} + b}{c\frac{e\alpha+f}{g\alpha+h} + d} = \frac{(ae + bg)\alpha + (af + bh)}{(ce + dg)\alpha + (cf + dh)}.$$

On the other hand,

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix},$$

so

$$(AB)\alpha = \frac{(ae + bg)\alpha + (af + bh)}{(ce + dg)\alpha + (cf + dh)}.$$

Therefore $A(B\alpha) = (AB)\alpha$, and the assertion follows. □

REMARK 2.4.3. By definition of the action, $A\alpha = (-A)\alpha$.

We now introduce unimodular equivalence of irrational numbers.

DEFINITION 2.4.4. Let α, β be irrational numbers. If there exists $A \in GL(2, \mathbb{Z})$ such that $\beta = A\alpha$, then α and β are said to be *unimodularly equivalent*, or simply *equivalent*. We write $\alpha \sim \beta$. The equivalence class

$$O_\alpha = \{\beta \mid \beta \sim \alpha\}$$

is called the *unimodular orbit*, or simply the *orbit*, of α.

The relation between the unimodular action on irrational numbers and continued fractions can be expressed succinctly using continued-fraction matrices.

PROPOSITION 2.4.5. *Let $a_0, a_1, \ldots, a_k, \alpha \in \mathbb{R}$. Whenever both sides are defined,*

$$[a_0; a_1, \ldots, a_k, \alpha] = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \alpha.$$

PROOF. By Proposition 2.2.3 and Theorem 2.2.4,

$$[a_0; a_1, \ldots, a_k, \alpha] = \frac{\alpha p_k + p_{k-1}}{\alpha q_k + q_{k-1}} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \alpha = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \alpha.$$

□

The next theorem characterizes equivalence of two irrational numbers in terms of their infinite continued-fraction expansions.

THEOREM 2.4.6 (Serret's theorem). *Let*

$$\alpha = [a_0; a_1, \ldots], \qquad \beta = [b_0; b_1, \ldots]$$

be irrational numbers. Then $\alpha \sim \beta$ if and only if there exist $n, m \in \mathbb{Z}_{\geq 0}$ such that

$$[a_n; a_{n+1}, \ldots] = [b_m; b_{m+1}, \ldots].$$

In particular, by uniqueness of regular continued-fraction expansions, $a_{n+h} = b_{m+h}$ for every $h \in \mathbb{Z}_{\geq 0}$.

We first prove a lemma.

LEMMA 2.4.7. *Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathbb{Z})$ and assume $c > d > 0$. Then there exist $\ell \in \mathbb{Z}_{\geq 0}$ and integers $c_0 \in \mathbb{Z}$, $c_1, \ldots, c_\ell \in \mathbb{Z}_{\geq 1}$ such that*

$$A = \begin{bmatrix} c_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_\ell & 1 \\ 1 & 0 \end{bmatrix}.$$

PROOF. Take the finite continued-fraction expansion

$$\frac{a}{c} = [a_0; a_1, \ldots, a_k].$$

Let $\frac{p_i}{q_i}$ be its i-th convergent. Then

$$\begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}.$$

If necessary, choose the expansion so that $a_k \geq 2$. Then we may also view

$$[a_0; a_1, \ldots, a_k] = [a_0; a_1, \ldots, a_k - 1, 1].$$

If $\frac{p'_i}{q'_i}$ denotes the i-th convergent for this latter expression, then

$$\begin{bmatrix} p'_{k+1} & p'_k \\ q'_{k+1} & q'_k \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k - 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here $a_k - 1 \geq 1$.

Since $A \in GL(2, \mathbb{Z})$ and

$$\det \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} = (-1)^{k+1}, \qquad \det \begin{bmatrix} p'_{k+1} & p'_k \\ q'_{k+1} & q'_k \end{bmatrix} = (-1)^{k+2},$$

exactly one of the following equalities holds:

$$\det A = \det \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}, \qquad \det A = \det \begin{bmatrix} p'_{k+1} & p'_k \\ q'_{k+1} & q'_k \end{bmatrix}.$$

We show that, in the first case,

$$A = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix},$$

and in the second case the analogous primed equality holds. We prove the first case; the second is identical.

Since $A \in GL(2, \mathbb{Z})$, the integers a and c are relatively prime. Since $c > 0$, the fraction $\frac{a}{c}$ is reduced. By Corollary 2.2.6, $\frac{p_k}{q_k}$ is also reduced. Hence $a = p_k$ and $c = q_k$. Using the equality of determinants gives $p_k d - b q_k = p_k q_{k-1} - q_k p_{k-1}$. Equivalently, $p_k(d - q_{k-1}) = q_k(b - p_{k-1})$. Since p_k and q_k are relatively prime, we have $q_k \mid d - q_{k-1}$. If $d - q_{k-1} \geq 0$, then the assumption $d < c$ gives $d - q_{k-1} < c - q_{k-1} = q_k - q_{k-1} < q_k$. If $d - q_{k-1} < 0$, then $q_{k-1} - d < q_{k-1} < q_k$. Thus in either case $|d - q_{k-1}| < q_k$. Since q_k divides $d - q_{k-1}$, we must have $d = q_{k-1}$. The equality $b = p_{k-1}$ then follows from the other three entries and the determinant equality. Hence

$$A = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}.$$

In the second determinant case, the same argument gives

$$A = \begin{bmatrix} p'_{k+1} & p'_k \\ q'_{k+1} & q'_k \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k - 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This proves the lemma. □

PROOF OF THEOREM 2.4.6. First assume that there exist $m, n \in \mathbb{Z}_{\geq 0}$ such that

$$[a_n; a_{n+1}, \ldots] = [b_m; b_{m+1}, \ldots].$$

Put

$$\alpha_n := [a_n; a_{n+1}, \ldots], \qquad \beta_m := [b_m; b_{m+1}, \ldots].$$

Then

$$\alpha = [a_0; a_1, \ldots, a_{n-1}, \alpha_n], \qquad \beta = [b_0; b_1, \ldots, b_{m-1}, \beta_m],$$

where, if $n = 0$, the first expression simply means $\alpha = \alpha_0$. By Proposition 2.4.5,

$$\alpha = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \alpha_n,$$

and

$$\beta = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_{m-1} & 1 \\ 1 & 0 \end{bmatrix} \beta_m.$$

Since $\alpha_n = \beta_m$, we may solve the first equality for α_n and substitute into the second equality. We obtain

$$\beta = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_{m-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix}^{-1} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \alpha.$$

Each factor lies in $GL(2, \mathbb{Z})$. Hence the product also lies in $GL(2, \mathbb{Z})$, and therefore α and β are equivalent.

Conversely, assume $\alpha \sim \beta$. Then there exists $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathbb{Z})$ such that $\beta = A\alpha$. Since $A\alpha = (-A)\alpha$, we may replace A by $-A$ if necessary and assume $c\alpha + d > 0$. For any $n \geq 1$,

$$\beta = A\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} \alpha_n,$$

where

$$\begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} := \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix}.$$

Set

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix}.$$

We show that n can be chosen so that $c' > d' > 0$. Direct computation gives

$$c' = cp_{n-1} + dq_{n-1} = q_{n-1} \left(\frac{cp_{n-1}}{q_{n-1}} + d \right), \tag{2.4.1}$$

$$d' = cp_{n-2} + dq_{n-2} = q_{n-2} \left(\frac{cp_{n-2}}{q_{n-2}} + d \right). \tag{2.4.2}$$

If $c = 0$, then after replacing A by $-A$ if necessary, we may assume $d > 0$. Then $c' = dq_{n-1}$ and $d' = dq_{n-2}$, so for sufficiently large n we have $q_{n-1} > q_{n-2} > 0$, and hence $c' > d' > 0$.

Thus assume $c \neq 0$. Since

$$\lim_{n\to\infty} \left(\frac{cp_{n-1}}{q_{n-1}} + d \right) = \lim_{n\to\infty} \left(\frac{cp_{n-2}}{q_{n-2}} + d \right) = c\alpha + d > 0,$$

we may take n sufficiently large so that both factors in parentheses in (2.4.1) and (2.4.2) are positive. Then $c', d' > 0$. Moreover,

$$\begin{aligned} c' - d' &= q_{n-1} \left(\frac{cp_{n-1}}{q_{n-1}} + d \right) - q_{n-2} \left(\frac{cp_{n-2}}{q_{n-2}} + d \right) > q_{n-1} \left(\frac{cp_{n-1}}{q_{n-1}} + d \right) - q_{n-1} \left(\frac{cp_{n-2}}{q_{n-2}} + d \right) \\ &= \frac{c}{q_{n-2}} (p_{n-1}q_{n-2} - q_{n-1}p_{n-2}) = \frac{(-1)^n c}{q_{n-2}}. \end{aligned}$$

Since n may still be chosen with either parity, we choose it so that $(-1)^n c > 0$. Then $c' - d' > 0$. Thus for such an n we have $c' > d' > 0$.

By Lemma 2.4.7, there exist $\ell \in \mathbb{Z}_{\geq 0}$ and $c_0 \in \mathbb{Z}$, $c_1, \ldots, c_\ell \in \mathbb{Z}_{\geq 1}$ such that

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} c_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_\ell & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence

$$\beta = \begin{bmatrix} c_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_\ell & 1 \\ 1 & 0 \end{bmatrix} \alpha_n = [c_0; c_1, \ldots, c_\ell, \alpha_n].$$

Since $\alpha_n = [a_n; a_{n+1}, \ldots]$, this is

$$\beta = [c_0; c_1, \ldots, c_\ell, a_n, a_{n+1}, \ldots].$$

Because $n \geq 1$, we have $a_n \geq 1$, so this is the infinite regular continued-fraction expansion of β. Taking $m = \ell + 1$, we obtain

$$[a_n; a_{n+1}, \ldots] = [b_m; b_{m+1}, \ldots].$$

This completes the proof. ☐

5. Periodic Continued Fractions and Quadratic Irrationals

In this section we study the case where the partial quotients in the regular continued-fraction expansion of an irrational number eventually become periodic. We characterize this phenomenon in terms of a property of the irrational number itself. We begin by defining periodic continued fractions.

Definition 2.5.1. An infinite regular continued fraction whose partial quotients are eventually periodic, as in

$$[a_0; a_1, a_2, \dots, a_k, b_0, b_1, \dots, b_\ell, b_0, b_1, \dots, b_\ell, b_0, b_1, \dots],$$

is called a *periodic continued fraction.* We write it as

$$[a_0; a_1, a_2, \dots, a_k, \overline{b_0, b_1, \dots, b_\ell}].$$

The finite sequence $(b_0, \dots, b_\ell)$ is called the *period.* A periodic continued fraction whose periodic part starts at the beginning is called a *purely periodic continued fraction.*

Next we define quadratic irrationals.

Definition 2.5.2. An irrational number α is called a *quadratic irrational* if it is a root of a quadratic equation with integer coefficients. Equivalently, there exist $a \in \mathbb{Z}_{\geq 1}$ and $b, c \in \mathbb{Z}$ with $\gcd(a, b, c) = 1$ such that

$$\alpha = \frac{-b + \varepsilon\sqrt{D}}{2a},$$

where $D = b^2 - 4ac$, $\varepsilon \in \{1, -1\}$, $D > 0$, and D is not a square. In this case D is called the *discriminant* of α. For such a quadratic irrational α, its quadratic conjugate is denoted by α', namely

$$\alpha' = \frac{-b - \varepsilon\sqrt{D}}{2a}.$$

If $\alpha > 1$ and $-1 < \alpha' < 0$, then α is called a *reduced quadratic irrational.*

If α is reduced, then $\alpha > \alpha'$, and hence necessarily $\varepsilon = 1$. We also impose $a \geq 1$ and $\gcd(a, b, c) = 1$ in order to make the discriminant D a well-defined quantity associated with α, rather than depending on a nonprimitive multiple of a quadratic equation.

Let I_2 be the set of all quadratic irrationals, and let R_2 be the set of all reduced quadratic irrationals. For a positive nonsquare integer d, let $I_2(d)$ be the set of quadratic irrationals with discriminant d, and let $R_2(d)$ be the set of reduced quadratic irrationals with discriminant d.

The main theorem of this section is the following.

Theorem 2.5.3 (Lagrange's theorem). *The following statements hold.*

1. *The infinite regular continued-fraction expansion of an irrational number α is periodic if and only if α is a quadratic irrational.*
2. *The infinite regular continued-fraction expansion of an irrational number α is purely periodic if and only if α is a reduced quadratic irrational.*

We prepare for the proof.

Theorem 2.5.4. *The action of the unimodular group on $\mathbb{R} \setminus \mathbb{Q}$ restricts to an action on I_2, and for every positive nonsquare integer $d \in \mathbb{Z}_{>0}$ it restricts to an action on $I_2(d)$.*

Proof. It suffices to verify that $I_2(d)$ is preserved. Let $\alpha \in I_2(d)$, and let $ax^2 + bx + c$ be a quadratic polynomial having α as a root, with $\gcd(a, b, c) = 1$. Then the discriminant $b^2 - 4ac$ is the discriminant of α. Let $M = \left[\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right] \in GL(2, \mathbb{Z})$ and put $\beta = M\alpha$. By Proposition 2.4.2, β is irrational. Solving $\beta = M\alpha$ for α, we obtain

$$\alpha = M^{-1}\beta = \frac{v\beta - t}{-u\beta + s}.$$

Substituting

$$x = \alpha = \frac{v\beta - t}{-u\beta + s}$$

into $ax^2 + bx + c = 0$ and clearing denominators gives

$$(av^2 + cu^2 - buv)\beta^2 + (-2atv + bsv + but - 2csu)\beta + at^2 + cs^2 - bst = 0.$$

Thus β is a root of the quadratic polynomial

$$(av^2 + cu^2 - buv)x^2 + (-2atv + bsv + but - 2csu)x + at^2 + cs^2 - bst.$$

Put

$$A = av^2 + cu^2 - buv, \qquad B = -2atv + bsv + but - 2csu, \qquad C = at^2 + cs^2 - bst.$$

To see that $B^2 - 4AC$ is the discriminant of β, we check that $\gcd(A, B, C) = 1$ and $A \neq 0$. If $A < 0$, we multiply the polynomial by -1; this makes the leading coefficient positive and does not change the root or the discriminant. The condition $A \neq 0$ follows because β is irrational. Now suppose $e \mid A$, $e \mid B$, and $e \mid C$. Then

$$\begin{aligned} s^2A + suB + u^2C &= a(sv - tu)^2 = a, \\ 2stA + (sv + tu)B + 2uvC &= b(sv - tu)^2 = b, \\ t^2A + tvB + v^2C &= c(sv - tu)^2 = c. \end{aligned}$$

Since $\gcd(a, b, c) = 1$, any common divisor of A, B, C must divide 1. Thus $\gcd(A, B, C) = 1$. Finally,

$$B^2 - 4AC = (b^2 - 4ac)(sv - tu)^2 = b^2 - 4ac = d,$$

because $M \in GL(2, \mathbb{Z})$. Hence $\beta \in I_2(d)$. □

THEOREM 2.5.5. *Fix a positive nonsquare integer d. For every*

$$\alpha = [a_0; a_1, \dots] \in I_2(d),$$

one has $\alpha_n := [a_n; a_{n+1}, \dots] \in R_2(d)$ for a sufficiently large n. Moreover, once this holds for some n, it holds for every $m \geq n$.

PROOF. Let

$$\alpha = [a_0; a_1, \dots] \in I_2(d), \qquad \alpha_k := [a_k; a_{k+1}, \dots].$$

For $k \geq 1$,

$$\alpha = [a_0; a_1, \dots, a_{k-1}, \alpha_k] = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{k-1} & 1 \\ 1 & 0 \end{bmatrix} \alpha_k,$$

where the last equality follows from Proposition 2.4.5. Since each matrix $\left[\begin{smallmatrix} a_i & 1 \\ 1 & 0 \end{smallmatrix}\right]$ lies in $GL(2, \mathbb{Z})$, the numbers α and α_k are equivalent. By Theorem 2.5.4, we obtain $\alpha_k \in I_2(d)$ for every $k \geq 0$.

Moreover, the construction algorithm (2.3.1) gives

$$\alpha_{k+1} = \frac{1}{\alpha_k - \lfloor \alpha_k \rfloor}.$$

Since $0 < \alpha_k - \lfloor \alpha_k \rfloor < 1$, we have $\alpha_{k+1} > 1$. Thus $\alpha_k > 1$ for all $k \geq 1$. It remains to show that, for all sufficiently large indices, the conjugate lies in the interval $(-1, 0)$.

From

$$\alpha = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \alpha_{k+1},$$

we solve for α_{k+1} and obtain

$$\alpha_{k+1} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}^{-1} \alpha = \begin{bmatrix} q_{k-1} & -p_{k-1} \\ -q_k & p_k \end{bmatrix} \alpha.$$

The inverse matrix may differ from the displayed matrix by an overall sign, but multiplying the matrix by -1 does not change the fractional linear transformation. Taking quadratic conjugates of both sides gives

$$\alpha'_{k+1} = -\frac{q_{k-1}\alpha' - p_{k-1}}{q_k\alpha' - p_k} = -\frac{q_{k-1}}{q_k}\frac{\alpha' - \frac{p_{k-1}}{q_{k-1}}}{\alpha' - \frac{p_k}{q_k}}.$$

Since $\lim_{k\to\infty} \frac{p_k}{q_k} = \alpha$, we have

$$\lim_{k\to\infty} \frac{\alpha' - \frac{p_{k-1}}{q_{k-1}}}{\alpha' - \frac{p_k}{q_k}} = 1.$$

Therefore, there exists N such that, for every $k \geq N$,

$$\frac{\alpha' - \frac{p_{k-1}}{q_{k-1}}}{\alpha' - \frac{p_k}{q_k}} > 0.$$

Since $q_{k-1}/q_k > 0$, this implies $\alpha'_{k+1} < 0$ for every $k \geq N$. Now take conjugates in the defining relation

$$\alpha_{k+2} = \frac{1}{\alpha_{k+1} - a_{k+1}}.$$

We obtain

$$\alpha'_{k+2} = -\frac{1}{a_{k+1} - \alpha'_{k+1}}.$$

Because $\alpha'_{k+1} < 0$ and $a_{k+1} \geq 1$, this gives

$$-1 < \alpha'_{k+2} < 0$$

for every $k \geq N$. Thus $\alpha_{k+2} \in R_2(d)$ for every $k \geq N$. Moreover, if α_m is reduced, then $\alpha_m > 1$ and $-1 < \alpha'_m < 0$ imply

$$\alpha_{m+1} = \frac{1}{\alpha_m - a_m} > 1, \qquad \alpha'_{m+1} = \frac{1}{\alpha'_m - a_m} \in (-1, 0),$$

so the reduced condition persists for all later indices. □

The preceding theorem immediately gives the following corollary.

COROLLARY 2.5.6. *For every positive nonsquare integer d and every $\alpha \in I_2(d)$, there exists $\beta \in R_2(d)$ such that $\alpha \sim \beta$. Thus every orbit in $I_2(d)$ has a representative in $R_2(d)$.*

LEMMA 2.5.7. *Let $\alpha \in R_2(d)$. Suppose that α is a root of $ax^2 + bx + c$, where $a \in \mathbb{Z}_{\geq 1}$ and $\gcd(a, b, c) = 1$. Then $0 < -b < \sqrt{d}$. In particular, $R_2(d)$ is finite for every positive nonsquare integer d.*

PROOF. Since α is reduced,

$$\alpha = \frac{-b + \sqrt{d}}{2a} > 1, \qquad -1 < \alpha' = \frac{-b - \sqrt{d}}{2a} < 0.$$

Multiplying by $2a$ gives

$$-b + \sqrt{d} > 2a > b + \sqrt{d} > 0.$$

In particular, $-b + \sqrt{d} > b + \sqrt{d}$ gives $0 < -b$, and $b + \sqrt{d} > 0$ gives $-b < \sqrt{d}$. Thus $0 < -b < \sqrt{d}$. For fixed d, there are only finitely many possible values of b. Since $d - b^2 = -4ac$ and $a, c \in \mathbb{Z}$, there are also only finitely many possible pairs (a, c). Hence $R_2(d)$ is finite. □

LEMMA 2.5.8. *Let α be a reduced quadratic irrational. If the continued fraction algorithm gives*

$$\alpha = [a_0; \alpha_1],$$

then

$$\lfloor \alpha \rfloor = a_0 = \left\lfloor -\frac{1}{\alpha'_1} \right\rfloor.$$

PROOF. From

$$\alpha = [a_0; \alpha_1] = a_0 + \frac{1}{\alpha_1},$$

taking conjugates gives

$$\alpha' = a_0 + \frac{1}{\alpha_1'}.$$

Rearranging, we obtain

$$-\frac{1}{\alpha_1'} = a_0 + (-\alpha').$$

Since $-1 < \alpha' < 0$, the integer part of $-\frac{1}{\alpha_1'}$ is a_0. □

PROOF OF THEOREM 2.5.3. We first prove the implication in (2) that a purely periodic continued fraction represents an element of R_2. Let

$$\alpha = [\overline{a_0; a_1, \ldots, a_{n-1}}]$$

with $n \geq 1$. Then

$$\alpha = [a_0; a_1, \ldots, a_{n-1}, \alpha] = \frac{\alpha p_{n-1} + p_{n-2}}{\alpha q_{n-1} + q_{n-2}}.$$

Rearranging gives

$$q_{n-1}\alpha^2 + (q_{n-2} - p_{n-1})\alpha - p_{n-2} = 0.$$

Thus α is a root of a quadratic equation. Since $\alpha \notin \mathbb{Q}$, we have $\alpha \in I_2$. Since the continued fraction is purely periodic,

$$\alpha = [a_0; a_1, \ldots, a_{n-1}, \alpha] = [a_0; a_1, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{n-1}, \alpha] = \cdots .$$

Thus $\alpha = \alpha_{kn}$ for every $k \in \mathbb{Z}_{\geq 0}$. By Theorem 2.5.5, this implies $\alpha \in R_2$.

Next we prove the implication in (1) that a periodic continued fraction represents an element of I_2. Suppose

$$\alpha = [a_0; a_1, a_2, \ldots, a_{n-1}, \overline{a_n, a_{n+1}, \ldots, a_{n+k-1}}]$$

for some $n \geq 1$ and $k \geq 1$. Put

$$\alpha_n = [\overline{a_n; a_{n+1}, \ldots, a_{n+k-1}}].$$

By the previous paragraph, $\alpha_n \in R_2 \subset I_2$. Since

$$\alpha = [a_0; a_1, a_2, \ldots, a_{n-1}, \alpha_n] = \begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} \alpha_n$$

and the matrix lies in $GL(2, \mathbb{Z})$, Theorem 2.5.4 shows that $\alpha \in I_2$.

We now prove the converse implication in (1): if $\alpha \in I_2$, then the continued-fraction expansion of α is periodic. Let d be the discriminant of α. Then d is a positive nonsquare integer and $\alpha \in I_2(d)$. By Theorem 2.5.5, for a sufficiently large n, $m \geq n$ implies $\alpha_m \in R_2(d)$. Since $R_2(d)$ is finite by Lemma 2.5.7, there exist indices

$$n \leq \ell < \ell'$$

such that $\alpha_\ell = \alpha_{\ell'}$. Then

$$\alpha_\ell = [a_\ell; a_{\ell+1}, \ldots, a_{\ell'-1}, \alpha_{\ell'}] = [a_\ell; a_{\ell+1}, \ldots, a_{\ell'-1}, \alpha_\ell].$$

Therefore

$$\alpha_\ell = [\overline{a_\ell; a_{\ell+1}, \ldots, a_{\ell'-1}}],$$

and the continued-fraction expansion of α is periodic.

Finally, we prove the converse implication in (2): if $\alpha \in R_2$, then the continued-fraction expansion of α is purely periodic. Let d be the discriminant of α, so $\alpha \in R_2(d)$. The Gauss map sends a reduced quadratic irrational to another reduced quadratic irrational with the same discriminant. Indeed, if $a_0 = \lfloor \alpha \rfloor$, then $\alpha > 1$ and $-1 < \alpha' < 0$ imply

$$\alpha_1 = \frac{1}{\alpha - a_0} > 1, \qquad \alpha_1' = \frac{1}{\alpha' - a_0} \in (-1, 0),$$

and the discriminant is preserved by the $GL(2,\mathbb{Z})$-action. Hence $\alpha_n \in R_2(d)$ for every $n \geq 0$. Since $R_2(d)$ is finite, there exist $0 \leq \ell < \ell'$ such that $\alpha_\ell = \alpha_{\ell'}$. If $\ell = 0$, then $\alpha_0 = \alpha_{\ell'}$, and the same argument as above gives pure periodicity.

Assume $\ell \geq 1$. Then

$$\alpha_{\ell-1} = [a_{\ell-1}; \alpha_\ell], \qquad \alpha_{\ell'-1} = [a_{\ell'-1}; \alpha_{\ell'}].$$

By Lemma 2.5.8,

$$a_{\ell-1} = \left\lfloor -\frac{1}{\alpha'_\ell} \right\rfloor = \left\lfloor -\frac{1}{\alpha'_{\ell'}} \right\rfloor = a_{\ell'-1}.$$

Thus $\alpha_{\ell-1} = \alpha_{\ell'-1}$. Repeating this step, we obtain

$$\alpha_0 = \alpha_{\ell'-\ell}.$$

The same argument as before then gives

$$\alpha = [\overline{a_0; a_1, \ldots, a_{\ell'-\ell-1}}],$$

so the expansion is purely periodic. □

Remark 2.5.9. In the proof that $\alpha \in R_2$ implies pure periodicity, one cannot simply assert at the outset that there exists ℓ with $\alpha_0 = \alpha_\ell$. The finiteness of $R_2(d)$ alone does not force α_0 to repeat. A priori, it could happen that α_0 is distinct from every later α_i, while the later α_i take only finitely many values. Lemma 2.5.8 is used precisely to rule out this possibility.

We end by recording a relation between a reduced quadratic irrational and the quadratic irrational obtained by reversing the period. This fact will be needed in later chapters.

Proposition 2.5.10. *If, for some $n \geq 1$,*

$$\alpha = [\overline{a_0; a_1, \ldots, a_{n-1}}] \in R_2,$$

then

$$-\frac{1}{\alpha'} = [\overline{a_{n-1}; a_{n-2}, \ldots, a_0}],$$

where α' is the quadratic conjugate of α.

Proof. From $\alpha = [\overline{a_0; a_1, \ldots, a_{n-1}}]$, we have

$$\alpha = [a_0; a_1, \ldots, a_{n-1}, \alpha] = \begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} \alpha = \frac{p_{n-1}\alpha + p_{n-2}}{q_{n-1}\alpha + q_{n-2}}.$$

Thus α is a root of

$$q_{n-1}x^2 + (q_{n-2} - p_{n-1})x - p_{n-2} = 0. \tag{2.5.1}$$

On the other hand,

$$\begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix}.$$

Taking transposes gives

$$\begin{bmatrix} p_{n-1} & q_{n-1} \\ p_{n-2} & q_{n-2} \end{bmatrix} = \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-2} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Put

$$\beta := [\overline{a_{n-1}; a_{n-2}, \ldots, a_0}].$$

Then

$$\beta = [a_{n-1}; a_{n-2}, \ldots, a_0, \beta] = \begin{bmatrix} p_{n-1} & q_{n-1} \\ p_{n-2} & q_{n-2} \end{bmatrix} \beta = \frac{p_{n-1}\beta + q_{n-1}}{p_{n-2}\beta + q_{n-2}}.$$

Rearranging, we obtain

$$p_{n-2}\beta^2 + (q_{n-2} - p_{n-1})\beta - q_{n-1} = 0.$$

Dividing by $-\beta^2$ gives

$$-p_{n-2} + (q_{n-2} - p_{n-1})\left(-\frac{1}{\beta}\right) + q_{n-1}\left(-\frac{1}{\beta}\right)^2 = 0.$$

Thus $-\frac{1}{\beta}$ is a root of (2.5.1). Since $a_{n-1} \geq 1$, Lemma 2.3.5 gives

$$\beta > \lfloor \beta \rfloor = a_{n-1} \geq 1,$$

and hence

$$-1 < -\frac{1}{\beta} < 0.$$

Therefore the conjugate root is $\alpha' = -\frac{1}{\beta}$. Equivalently,

$$\beta = -\frac{1}{\alpha'}.$$

This proves the claim. □

CHAPTER 3

Lagrange Spectrum

Since Chapter 2 prepared the basic facts on continued fractions and quadratic irrationals, we now use them in this chapter to study the fundamental properties of the Lagrange spectrum. The Lagrange spectrum is the set of all Lagrange constants, which measure how well irrational numbers can be approximated by rational numbers, and it is one of the most fundamental objects in Diophantine approximation theory. Looking only at the definition, one might get no more than the impression that it is a multiplicative analogue of the irrationality exponent, which measures the quality of approximation in terms of powers of the denominator. However, when viewed through continued fraction expansions, one sees that its values are deeply connected with infinite sequences and periodicity.

We first define the Lagrange constant, give a computable expression for it, and compute basic examples. We then introduce a description in terms of bi-infinite sequences, which makes it easier to compare the Lagrange spectrum with the Markov spectrum in the next chapter. Finally, for quadratic irrationals, we show that the action of $GL(2,\mathbb{Z})$ and the theory of periodic continued fractions developed in the previous chapter make it possible to compute the Lagrange constant explicitly.

Standard texts that are written with Markov's theorem in mind, such as [**Aig13, Bom07, Reu19**], often impose the assumption that the Lagrange constant is at most 3 from the beginning. In this chapter, however, we do not impose such an assumption and work in the general setting.

The discussion in this chapter follows [**Aig13**].

1. Definitions and First Examples

In this section we introduce the Lagrange spectrum and check the simplest examples.

Definition 3.1.1. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. We define $\mathcal{L}(\alpha)$ to be the supremum of all positive real numbers L satisfying the following condition:

- there exist infinitely many reduced fractions p/q with $q>0$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}.$$

If this set is unbounded above, we set $\mathcal{L}(\alpha)=\infty$. We call $\mathcal{L}(\alpha)$ the *Lagrange constant* of α. The set

$$\mathcal{L} := \{\mathcal{L}(\alpha) \mid \alpha \in \mathbb{R}\setminus\mathbb{Q}\}$$

is called the *Lagrange spectrum*.

Although rational numbers are not included in the definition, it is useful to first see what would happen if the same condition were applied to a rational number.

Proposition 3.1.2. *Let α be a rational number, and let L be any positive real number. Then there are only finitely many reduced fractions p/q with $q>0$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}.$$

Proof. There is at most one reduced fraction p/q equal to α, so we assume below that $p/q \neq \alpha$. Write $\alpha = a/b$, where a/b is reduced. Then

$$\left|\alpha - \frac{p}{q}\right| = \frac{|aq-bp|}{bq} \geq \frac{1}{bq}.$$

Thus, if $|\alpha - p/q| < 1/(Lq^2)$, then $1/(Lq^2) > 1/(bq)$, and hence $q < b/L$. Since q is a positive integer, there are only finitely many possible values of q. For each fixed such q, solving the same inequality for p gives

$$\frac{aq}{b} - \frac{1}{Lq} < p < \frac{aq}{b} + \frac{1}{Lq}.$$

Since p is an integer, there are only finitely many such p for each fixed q. Hence only finitely many reduced fractions satisfy the inequality. □

The proposition shows that, for a rational number α, the set of positive real numbers L satisfying the condition in Definition 3.1.1 is empty. Thus rational numbers do not lead to a meaningful Lagrange constant in this sense. In the rest of this section we consider irrational numbers. We first give a characterization of the Lagrange constant in terms of continued fractions.

THEOREM 3.1.3. *Let α be an irrational number with infinite continued-fraction expansion $\alpha = [a_0; a_1, \ldots]$. Put*

$$\alpha_n := [a_n; a_{n+1}, \ldots], \qquad \beta_n := [a_n; a_{n-1}, \ldots, a_1].$$

Then

$$\mathcal{L}(\alpha) = \limsup_{n\to\infty} \left(\alpha_{n+1} + \frac{1}{\beta_n}\right). \tag{3.1.1}$$

In what follows we write

$$\lambda_n(\alpha) := \alpha_{n+1} + \frac{1}{\beta_n}.$$

LEMMA 3.1.4. *Let $\alpha = [a_0; a_1, \ldots]$ be the infinite continued-fraction expansion of an irrational number. Then, for every $k \geq 1$,*

$$\frac{q_k}{q_{k-1}} = [a_k; a_{k-1}, \ldots, a_1] (= \beta_k).$$

PROOF. We prove this by induction. For $k = 1$, we have $q_1/q_0 = a_1$. Assume the claim holds for $k - 1$. Then

$$\frac{q_k}{q_{k-1}} = \frac{a_k q_{k-1} + q_{k-2}}{q_{k-1}} = a_k + [0; a_{k-1}, \ldots, a_1] = [a_k; a_{k-1}, \ldots, a_1].$$

This proves the claim for k. □

PROOF OF THEOREM 3.1.3. For the convergents of α we have

$$\left|\alpha - \frac{p_n}{q_n}\right| = \left|\frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{1}{\alpha_{n+1}q_n^2 + q_{n-1}q_n} = \frac{1}{\lambda_n(\alpha)q_n^2},$$

where the last equality uses Lemma 3.1.4.

We first show that $\mathcal{L}(\alpha) \leq \limsup_{n\to\infty} \lambda_n(\alpha)$. Let L be a positive real number such that there exist infinitely many reduced fractions p/q satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}.$$

If $L > 2$, then by Theorem 2.3.10 all these reduced fractions are convergents. Hence for infinitely many n we have

$$\frac{1}{\lambda_n(\alpha)q_n^2} < \frac{1}{Lq_n^2},$$

and therefore $\lambda_n(\alpha) > L$ for infinitely many n. Thus $L \leq \limsup_{n\to\infty} \lambda_n(\alpha)$.

If $L \leq 2$, the same conclusion follows from $\limsup_{n\to\infty} \lambda_n(\alpha) \geq 2$. Indeed, if partial quotients $a_j \geq 2$ occur infinitely often, then $\lambda_{j-1}(\alpha) > 2$ occurs infinitely often. Otherwise, for all sufficiently large j we have $a_j = 1$, and in this case

$$\limsup_{j\to\infty} \lambda_j(\alpha) = \lim_{j\to\infty} \lambda_j(\alpha) = \sqrt{5} > 2.$$

Thus in every case $L \le \limsup_{n\to\infty} \lambda_n(\alpha)$. Taking the supremum over all such L gives

$$\mathcal{L}(\alpha) \le \limsup_{n\to\infty} \lambda_n(\alpha).$$

Conversely, for any $\varepsilon > 0$, by the definition of the limit superior there exist infinitely many n such that

$$\lambda_n(\alpha) > \limsup_{m\to\infty} \lambda_m(\alpha) - \varepsilon.$$

For these n,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{(\limsup_{m\to\infty} \lambda_m(\alpha) - \varepsilon)q_n^2}.$$

Hence

$$\limsup_{n\to\infty} \lambda_n(\alpha) - \varepsilon \le \mathcal{L}(\alpha).$$

Since $\varepsilon > 0$ was arbitrary, we also get

$$\limsup_{n\to\infty} \lambda_n(\alpha) \le \mathcal{L}(\alpha).$$

This proves the theorem. □

Example 3.1.5. Let us compute some examples using Theorem 3.1.3.

(1) Let $\alpha = (1+\sqrt{5})/2$. Since $\frac{1+\sqrt{5}}{2} = [1; \frac{1+\sqrt{5}}{2}]$, we have $\frac{1+\sqrt{5}}{2} = [\overline{1}]$. Therefore

$$\mathcal{L}\left(\frac{1+\sqrt{5}}{2}\right) = \limsup_{n\to\infty}([\overline{1}] + [0; 1, \dots, 1]) = [\overline{1}] + \lim_{n\to\infty}[0; 1, \dots, 1] = [\overline{1}] + [0; \overline{1}] = \sqrt{5}.$$

(2) Let $\alpha = 1+\sqrt{2}$. Since $1+\sqrt{2} = [2; 1+\sqrt{2}]$, we have $1+\sqrt{2} = [\overline{2}]$. Therefore

$$\mathcal{L}(1+\sqrt{2}) = \limsup_{n\to\infty}([\overline{2}] + [0; 2, \dots, 2]) = [\overline{2}] + \lim_{n\to\infty}[0; 2, \dots, 2] = [\overline{2}] + [0; \overline{2}] = 2\sqrt{2}.$$

We finish this section by observing that the Lagrange spectrum can be described using bi-infinite sequences and a limit superior. Let $\mathbf{a} = (\dots, a_{-1}, a_0, a_1, \dots)$ be a bi-infinite sequence with $a_i \in \mathbb{Z}_{\ge 1}$ for every $i \in \mathbb{Z}$. Define

$$\ell_n(\mathbf{a}) := [a_n; a_{n+1}, \dots] + [0; a_{n-1}, a_{n-2}, \dots].$$

Then the following result holds.

Corollary 3.1.6 (Perron's Identity). *The Lagrange spectrum is characterized as*

$$\mathcal{L} = \left\{ \limsup_{n\to+\infty} \ell_n(\mathbf{a}) \;\middle|\; \mathbf{a} \in \mathbb{Z}_{\ge 1}^{\mathbb{Z}} \right\}.$$

We begin with the following lemma.

Lemma 3.1.7. *Let x and y be real numbers whose continued-fraction expansions have a common initial block*

$$[a_0; a_1, \dots, a_{n-1}]$$

of length $n \ge 2$. If one of x and y is rational, we use a finite continued-fraction expansion having this displayed initial block. Then

$$|x - y| \le \frac{1}{n(n-1)}.$$

Proof. Let

$$\frac{p_j}{q_j} := [a_0; a_1, \dots, a_j]$$

be the convergents of the common initial block. Any real number whose continued-fraction expansion begins with $a_0, \dots, a_{n-1}$ can be written in the form

$$[a_0; a_1, \dots, a_{n-1}, t]$$

with $1 \le t \le \infty$, where $t = \infty$ corresponds to the finite continued fraction $[a_0; \dots, a_{n-1}]$. By Proposition 2.4.5, this is

$$\frac{tp_{n-1} + p_{n-2}}{tq_{n-1} + q_{n-2}}.$$

As t ranges over $[1,\infty]$, these values lie in an interval whose endpoints are

$$\frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}} \quad \text{and} \quad \frac{p_{n-1}}{q_{n-1}}.$$

The length of this interval is

$$\frac{1}{q_{n-1}(q_{n-1}+q_{n-2})}.$$

Since $q_{n-1} \geq n-1$ and $q_{n-1}+q_{n-2} \geq n$, this length is at most $1/(n(n-1))$. Hence $|x-y| \leq 1/(n(n-1))$. □

Proof of Corollary 3.1.6. Let the set on the right-hand side be

$$\mathcal{R} := \left\{ \limsup_{n\to+\infty} \ell_n(\mathbf{a}) \;\middle|\; \mathbf{a} \in \mathbb{Z}_{\geq 1}^{\mathbb{Z}} \right\}.$$

We prove $\mathcal{R} = \mathcal{L}$.

First we show $\mathcal{R} \subset \mathcal{L}$. Take an arbitrary

$$\mathbf{a} = (\ldots, a_{-1}, a_0, a_1, \ldots) \in \mathbb{Z}_{\geq 1}^{\mathbb{Z}}$$

and put

$$\alpha := [0; a_1, a_2, \ldots].$$

Set

$$u_n := [0; a_{n-1}, a_{n-2}, \ldots, a_1], \qquad v_n := [0; a_{n-1}, a_{n-2}, \ldots, a_1, a_0, a_{-1}, \ldots].$$

By Theorem 3.1.3,

$$\mathcal{L}(\alpha) = \limsup_{n\to\infty} \left([a_n; a_{n+1}, a_{n+2}, \ldots] + u_n\right),$$

whereas

$$\ell_n(\mathbf{a}) = [a_n; a_{n+1}, a_{n+2}, \ldots] + v_n.$$

Here u_n and v_n have continued-fraction expansions with the same initial block of length n, namely $0, a_{n-1}, a_{n-2}, \ldots, a_1$. Applying Lemma 3.1.7, we obtain

$$|u_n - v_n| \leq \frac{1}{n(n-1)} \qquad (n \geq 2).$$

Therefore

$$|\ell_n(\mathbf{a}) - ([a_n; a_{n+1}, a_{n+2}, \ldots] + u_n)| \leq \frac{1}{n(n-1)} \to 0 \qquad (n \to \infty),$$

and so

$$\limsup_{n\to\infty} \ell_n(\mathbf{a}) = \mathcal{L}(\alpha) \in \mathcal{L}.$$

Thus $\mathcal{R} \subset \mathcal{L}$.

Next we show $\mathcal{L} \subset \mathcal{R}$. Take $r \in \mathcal{L}$. Then there is an irrational number

$$\alpha = [a_0; a_1, a_2, \ldots]$$

such that $r = \mathcal{L}(\alpha)$. Put

$$\widetilde{\alpha} := [0; a_1, a_2, \ldots].$$

Since the expression in (3.1.1) does not depend on a_0, we have

$$\mathcal{L}(\widetilde{\alpha}) = \mathcal{L}(\alpha) = r.$$

Define a bi-infinite sequence $\mathbf{b} = (b_n)_{n\in\mathbb{Z}}$ by

$$b_n = \begin{cases} a_n & (n \geq 1), \\ 1 & (n \leq 0). \end{cases}$$

Set

$$u_n := [0; a_{n-1}, a_{n-2}, \ldots, a_1], \qquad w_n := [0; a_{n-1}, a_{n-2}, \ldots, a_1, 1, 1, \ldots].$$

By Theorem 3.1.3,

$$\mathcal{L}(\widetilde{\alpha}) = \limsup_{n\to\infty} \left([a_n; a_{n+1}, a_{n+2}, \ldots] + u_n\right),$$

whereas
$$\ell_n(\mathbf{b}) = [a_n; a_{n+1}, a_{n+2}, \dots] + w_n.$$
Again u_n and w_n have the same initial block $0, a_{n-1}, a_{n-2}, \dots, a_1$ of length n, so Lemma 3.1.7 gives
$$|u_n - w_n| \leq \frac{1}{n(n-1)} \qquad (n \geq 2).$$
Hence
$$\limsup_{n\to\infty} \ell_n(\mathbf{b}) = \mathcal{L}(\tilde{\alpha}) = r.$$
Thus $r \in \mathcal{R}$, and $\mathcal{L} \subset \mathcal{R}$. This proves the desired equality. □

Corollary 3.1.6 was presented as Perron's characterization of $\mathcal{L}$, but for the purposes of this text it is not the most useful form. In its proof, the left-hand side of the bi-infinite sequence attached to an irrational number α was filled with 1's. Nothing essential depends on this choice: any sequence could have been placed on the left, and the same argument would still work. Thus Corollary 3.1.6 mainly reformulates Theorem 3.1.3. What will be more useful later is the formula (3.1.1) itself and the construction, from a Lagrange constant, of a bi-infinite sequence for which the relevant value is realized as a *supremum*. This is the subject of the next section.

2. A Supremum Construction from Bi-infinite Sequences

In this section we associate to finite Lagrange constants certain bi-infinite sequences for which the relevant value is realized as a supremum. Notice that here we use a supremum, not a limit superior. We begin with the following proposition.

PROPOSITION 3.2.1. *Let α be an irrational number with infinite continued-fraction expansion $\alpha = [a_0; a_1, \dots]$.*

(1) *If $(a_n)_{n=0}^{\infty}$ is bounded, then $\mathcal{L}(\alpha) < \infty$.*
(2) *If $(a_n)_{n=0}^{\infty}$ is unbounded, then $\mathcal{L}(\alpha) = \infty$.*

PROOF. For each $k \geq 2$, Lemmas 2.2.14 and 2.3.5 give
$$1 \leq a_k < \alpha_k + \frac{1}{\beta_{k-1}} < a_k + 2.$$
If the sequence $(a_n)_{n=0}^{\infty}$ is bounded and z is an upper bound for $(a_n)_{n=2}^{\infty}$, then
$$\alpha_n + \frac{1}{\beta_{n-1}} < z + 2$$
for every $n \geq 2$. Hence $(\lambda_n(\alpha))_{n=1}^{\infty}$ is bounded above, and Theorem 3.1.3 gives $\mathcal{L}(\alpha) < \infty$.

Conversely, suppose that $(a_n)_{n=0}^{\infty}$ is unbounded. Then the subsequence $(a_n)_{n=2}^{\infty}$ is unbounded after discarding finitely many initial terms. Since
$$\lambda_{n-1}(\alpha) = \alpha_n + \frac{1}{\beta_{n-1}} > a_n,$$
the sequence $(\lambda_n(\alpha))_{n=1}^{\infty}$ is unbounded along arbitrarily large indices. Therefore its limit superior is infinite, and Theorem 3.1.3 gives $\mathcal{L}(\alpha) = \infty$. □

It follows that $\mathcal{L}(\alpha)$ is a meaningful finite number only when the partial quotients of α are bounded. Thus, when studying finite elements of the Lagrange spectrum, we may assume that $(a_n)_{n=0}^{\infty}$ is bounded.

PROPOSITION 3.2.2. *Let α be an irrational number with infinite continued-fraction expansion $\alpha = [a_0; a_1, \dots]$, and assume that $(a_n)_{n=0}^{\infty}$ is bounded. Put*
$$\alpha_n = [a_n; a_{n+1}, \dots], \qquad \beta_n = [a_n; a_{n-1}, \dots, a_1], \qquad \lambda_n(\alpha) = \alpha_{n+1} + \frac{1}{\beta_n}.$$
Then the following hold:

(1) *$(\alpha_n)_{n=0}^{\infty}$ is bounded.*
(2) *$(\beta_n)_{n=1}^{\infty}$ is bounded.*

(3) $(\lambda_n(\alpha))_{n=1}^{\infty}$ *is bounded.*

PROOF. Let a be the maximum of $(a_n)_{n=1}^{\infty}$. For (1), Lemma 2.3.5 gives $a_k < \alpha_k < a_k + 1$ for every $k \geq 1$, and hence $1 < \alpha_n < a + 1$ for every $n \geq 1$. Thus $(\alpha_n)_{n=0}^{\infty}$ is bounded. For (2), the same argument using Lemma 2.2.14 gives $a_k \leq \beta_k < a_k + 1$, so $1 \leq \beta_n < a + 1$ for every $n \geq 1$. Thus $(\beta_n)_{n=1}^{\infty}$ is bounded. Finally, (3) follows from (1) and (2), since

$$1 < \alpha_{n+1} + \frac{1}{\beta_n} < a + 2$$

for every $n \geq 1$. □

Since $(\alpha_{n+1} + 1/\beta_n)_{n=1}^{\infty}$ is bounded, the Bolzano–Weierstrass theorem gives a convergent subsequence $(\alpha_{n_i+1} + 1/\beta_{n_i})_{i=1}^{\infty}$. Let its limit be r. Since $(\beta_{n_i})_{i=1}^{\infty}$ is also bounded, it has a further convergent subsequence; write its limit as η. Along the same indices, the corresponding subsequence of $(\alpha_{n+1})_{n=1}^{\infty}$ converges to $r - 1/\eta$. Put

$$\theta := r - \frac{1}{\eta}.$$

The pair (θ, η) obtained in this way from the accumulation point r will be called a *pair associated with the accumulation point* r. By the definition of the limit superior, $\mathcal{L}(\alpha)$ is the supremum of the accumulation points r obtained from such subsequences. To avoid excessive subscripts, we shall henceforth denote the chosen subsequence simply by the indices n_i.

PROPOSITION 3.2.3. *Let α be an irrational number, and let $(a_n)_{n=0}^{\infty}$ be the sequence giving its infinite continued-fraction expansion. Assume that $(a_n)_{n=1}^{\infty}$ is bounded, and let a be the maximum of this sequence, excluding a_0. Let r be an accumulation point of $(\lambda_n(\alpha))_{n=1}^{\infty}$, and let (θ, η) be a pair associated with r. Then θ and η are irrational numbers. If*

$$\theta = [b_0; b_1, \dots], \qquad \eta = [b_{-1}; b_{-2}, \dots],$$

then

$$1 \leq b_i \leq a$$

for every $i \in \mathbb{Z}$. In particular, $1 < \theta, \eta < a + 1$.

PROOF. Choose indices n_i such that

$$\alpha_{n_i+1} \to \theta, \qquad \beta_{n_i} \to \eta.$$

We first prove that θ is irrational.

Suppose, to the contrary, that θ is rational, and write $\theta = p/q$ in lowest terms with $q > 0$. Since $\alpha_{n_i+1} \to \theta$, for all sufficiently large i we have

$$\left|\alpha_{n_i+1} - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

By Theorem 2.3.10, the rational number p/q is a convergent of α_{n_i+1} for all sufficiently large i.

Let

$$\theta = [c_0; c_1, \dots, c_m]$$

be one of the finite regular continued-fraction expansions of θ. Since a rational number has only two finite regular continued-fraction expansions, after passing to a subsequence we may assume that the same finite expansion occurs as the initial block of the continued-fraction expansion of α_{n_i+1} for every i. Thus we may write

$$\alpha_{n_i+1} = [c_0; c_1, \dots, c_m, \xi_i]$$

for some tail ξ_i. Here ξ_i is again a complete quotient of α, and hence its first partial quotient is one of the numbers a_n with $n \geq 1$. Therefore

$$1 < \xi_i < a + 1$$

for every i.

On the other hand, the function

$$\xi \longmapsto [c_0; c_1, \dots, c_m, \xi]$$

converges to $[c_0; c_1, \dots, c_m] = \theta$ only when $\xi \to \infty$. Equivalently, this follows from the formula

$$[c_0; c_1, \dots, c_m, \xi] = \frac{\xi p_m + p_{m-1}}{\xi q_m + q_{m-1}},$$

where p_j/q_j denotes the j-th convergent of $[c_0; c_1, \dots, c_m]$. Since the sequence (ξ_i) is bounded above by $a+1$, the values $[c_0; c_1, \dots, c_m, \xi_i]$ cannot converge to $[c_0; c_1, \dots, c_m]$. This contradicts $\alpha_{n_i+1} \to \theta$. Hence θ is irrational.

Write

$$\theta = [b_0; b_1, b_2, \dots].$$

We now show that all partial quotients b_j lie in $\{1, \dots, a\}$. Fix $k \geq 0$. Since θ is irrational, the condition that a real number have initial partial quotients

$$b_0, b_1, \dots, b_k$$

defines an open interval containing θ. Since $\alpha_{n_i+1} \to \theta$, it follows that, for all sufficiently large i, the continued-fraction expansion of α_{n_i+1} has the same initial block:

$$\alpha_{n_i+1} = [b_0; b_1, \dots, b_k, \dots].$$

Therefore

$$b_\ell = a_{n_i+\ell+1} \qquad (0 \leq \ell \leq k)$$

for all sufficiently large i. Since each a_n with $n \geq 1$ belongs to $\{1, \dots, a\}$, we obtain

$$1 \leq b_\ell \leq a \qquad (0 \leq \ell \leq k).$$

As k was arbitrary, this proves

$$1 \leq b_j \leq a \qquad (j \geq 0).$$

The proof for η is analogous. Suppose first that η is rational, and write $\eta = p/q$ in lowest terms. Since $\beta_{n_i} \to \eta$, for all sufficiently large i we have

$$\left| \beta_{n_i} - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

Applying Theorem 2.3.10 to the finite continued fraction β_{n_i}, we see that p/q is a convergent of β_{n_i} for all sufficiently large i. After passing to a subsequence, we may again assume that one fixed finite continued-fraction expansion

$$\eta = [d_0; d_1, \dots, d_m]$$

occurs as an initial block of β_{n_i}. Since $n_i \to \infty$, the tail after this block is defined for all sufficiently large i, and we can write

$$\beta_{n_i} = [d_0; d_1, \dots, d_m, \zeta_i].$$

The first partial quotient of ζ_i is again one of the numbers a_n with $n \geq 1$, so $1 \leq \zeta_i < a+1$. As above, convergence

$$[d_0; d_1, \dots, d_m, \zeta_i] \to [d_0; d_1, \dots, d_m]$$

would force $\zeta_i \to \infty$, contradicting the boundedness of (ζ_i). Hence η is irrational.

Write

$$\eta = [b_{-1}; b_{-2}, b_{-3}, \dots].$$

Fix $k \geq 1$. Since η is irrational, the condition that a real number have initial partial quotients

$$b_{-1}, b_{-2}, \dots, b_{-k}$$

defines an open interval containing η. Since $\beta_{n_i} \to \eta$, for all sufficiently large i we have

$$\beta_{n_i} = [b_{-1}; b_{-2}, \dots, b_{-k}, \dots].$$

On the other hand,

$$\beta_{n_i} = [a_{n_i}; a_{n_i-1}, \dots, a_1].$$

Thus

$$b_{-\ell} = a_{n_i-\ell+1} \qquad (1 \leq \ell \leq k)$$

for all sufficiently large i. Since all these a-entries belong to $\{1,\dots,a\}$, we obtain

$$1 \le b_{-\ell} \le a \qquad (1 \le \ell \le k).$$

As k was arbitrary, this proves

$$1 \le b_j \le a \qquad (j < 0).$$

Combining the two parts, we have

$$1 \le b_i \le a \qquad (i \in \mathbb{Z}).$$

Finally, since $\theta = [b_0; b_1, \dots]$ and $\eta = [b_{-1}; b_{-2}, \dots]$ are irrational with first partial quotients between 1 and a, we have

$$1 < \theta < a+1, \qquad 1 < \eta < a+1.$$

This proves the proposition. □

Let r be an accumulation point of $(\lambda_n(\alpha))_{n=1}^{\infty}$, and let (θ, η) be a pair associated with it. Write

$$\theta = [b_0; b_1, \dots], \qquad \eta = [b_{-1}; b_{-2}, \dots],$$

and consider the bi-infinite sequence

$$\mathbf{b} = (\dots, b_{-1}, b_0, b_1, \dots).$$

We call this the *bi-infinite sequence determined by* (θ, η). Then

$$\ell_0(\mathbf{b}) = [b_0; b_1, \dots] + [0; b_{-1}, b_{-2}, \dots] = \theta + \frac{1}{\eta} = r.$$

THEOREM 3.2.4. *Let α be an irrational number whose partial quotients are bounded, and consider the sequence $(\lambda_n(\alpha))_{n=1}^{\infty}$. Let r be any accumulation point of this sequence, and let*

$$\mathbf{b} = (\dots, b_{-2}, b_{-1}, b_0, b_1, \dots)$$

be the bi-infinite sequence determined by a pair (θ, η) associated with r. For any $h \in \mathbb{Z}$, put

$$\theta' := [b_h; b_{h+1}, \dots], \qquad \eta' := [b_{h-1}; b_{h-2}, \dots]$$

and define

$$r' := \ell_h(\mathbf{b}) = \theta' + \frac{1}{\eta'}.$$

Then r' is also an accumulation point of $(\lambda_n(\alpha))_{n=1}^{\infty}$.

PROOF. We prove the case $h > 0$. Since

$$\theta = [b_0; b_1, \dots, b_{h-1}, \theta'],$$

if p_k/q_k denotes the convergents of θ, then

$$\theta = \frac{\theta' p_{h-1} + p_{h-2}}{\theta' q_{h-1} + q_{h-2}}. \tag{3.2.1}$$

Choose indices n_i such that $\alpha_{n_i+1} \to \theta$ and $\beta_{n_i} \to \eta$. By the argument in the proof of Proposition 3.2.3, for all sufficiently large i the first h partial quotients of α_{n_i+1} agree with those of θ. Thus

$$\alpha_{n_i+1} = [b_0; b_1, \dots, b_{h-1}, \alpha_{n_i+h+1}],$$

and hence

$$\alpha_{n_i+1} = \frac{\alpha_{n_i+h+1} p_{h-1} + p_{h-2}}{\alpha_{n_i+h+1} q_{h-1} + q_{h-2}}. \tag{3.2.2}$$

Solving (3.2.1) for θ' and (3.2.2) for α_{n_i+h+1}, we obtain integers A, B, C, D such that

$$\theta' = \frac{\theta A + B}{\theta C + D}, \qquad \alpha_{n_i+h+1} = \frac{\alpha_{n_i+1} A + B}{\alpha_{n_i+1} C + D}.$$

Since $\alpha_{n_i+1} \to \theta$, it follows that $\alpha_{n_i+h+1} \to \theta'$.

For η we use

$$\eta' = [b_{h-1}; b_{h-2}, \dots, b_0, \eta], \qquad \beta_{n_i+h} = [b_{h-1}; b_{h-2}, \dots, b_0, \beta_{n_i}].$$

The same argument gives $\beta_{n_i+h} \to \eta'$. Therefore

$$\lambda_{n_i+h}(\alpha) \to \theta' + \frac{1}{\eta'} = r'.$$

Thus r' is an accumulation point. The case $h < 0$ is proved in the same way, with the roles of θ and η interchanged. ☐

Let $\mathscr{A}$ be the set of all bi-infinite sequences $(a_i)_{i=-\infty}^{\infty}$ with $a_i \in \mathbb{Z}_{\geq 1}$ for every $i \in \mathbb{Z}$. For $\mathbf{a} \in \mathscr{A}$, define

$$\mathcal{S}(\mathbf{a}) := \sup_{h \in \mathbb{Z}} \ell_h(\mathbf{a}),$$

and set $\mathcal{S}(\mathbf{a}) = \infty$ when the set is unbounded above. The preceding discussion gives the following theorem.

THEOREM 3.2.5. *Let*

$$\mathcal{S} := \{\mathcal{S}(\mathbf{a}) \mid \mathbf{a} \in \mathscr{A}\}.$$

Then $\mathcal{L} \subset \mathcal{S}$*.*

PROOF. For $\infty \in \mathcal{L}$, take a bi-infinite sequence $\mathbf{a}$ whose right-hand partial quotients a_h are unbounded. Since $\alpha_h = [a_h; a_{h+1}, \ldots]$ satisfies $\alpha_h > a_h$, we have

$$\sup_{h \in \mathbb{Z}} \ell_h(\mathbf{a}) \geq \sup_{h \in \mathbb{Z}} \alpha_h = \infty.$$

Thus $\infty \in \mathcal{S}$.

Now suppose $\mathcal{L}(\alpha) = r < \infty$. By the proposition above, the partial quotients of α are bounded. The number r can be taken as an accumulation point of $(\lambda_n(\alpha))_{n=1}^{\infty}$. Let $\mathbf{b} = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$ be the bi-infinite sequence determined by a pair associated with r. Then $\ell_0(\mathbf{b}) = r$. By Theorem 3.2.4, for every $h \in \mathbb{Z}$, the number $\ell_h(\mathbf{b})$ is an accumulation point of $(\lambda_n(\alpha))_{n=1}^{\infty}$. By Theorem 3.1.3, r is the supremum of all accumulation points of this sequence. Hence

$$r = \mathcal{S}(\mathbf{b}) \in \mathcal{S}.$$

This proves the assertion. ☐

We end this section with several cautions. In general, it is difficult to compute an accumulation point r of $(\lambda_n(\alpha))_{n=1}^{\infty}$ and a pair (θ, η) associated with it directly from the infinite continued-fraction expansion of an irrational number α. Moreover, the pair (θ, η) associated with r need not be unique, because there may be choices in taking η (or, if the order of the two limits is reversed, in taking θ). Thus a single accumulation point r may give rise to more than one bi-infinite sequence.

Theorem 3.2.4 says that once one fixes a bi-infinite sequence associated with a single accumulation point r, one can produce further accumulation points r' from it. However, it does not assert that all accumulation points of $(\lambda_n(\alpha))_{n=1}^{\infty}$ are obtained from that one bi-infinite sequence.

It might seem that Theorem 3.2.5 supplies this missing assertion, but it does not. That theorem only says that if one takes a bi-infinite sequence associated with the accumulation point which realizes the Lagrange constant, then the Lagrange constant is the supremum of the accumulation points obtained from that particular sequence. This is, in a sense, a rather formal consequence of the construction. Therefore, even if one finds a bi-infinite sequence $\mathbf{b}$ arising from an accumulation point r and an associated pair, it need not be true that $\mathcal{S}(\mathbf{b}) = \mathcal{L}(\alpha)$.

For these reasons, the material in this section alone does not provide a practical method for computing $\mathcal{L}(\alpha)$ for a general irrational number α.

3. Lagrange Constants of Quadratic Irrationals

At the end of the previous section we explained that several difficulties make it hard to compute the Lagrange constant of a general irrational number by means of bi-infinite sequences. However, when the infinite continued-fraction expansion of α has a particularly simple form, these difficulties can be overcome. The simple form in question is pure periodicity, or equivalently, the case where α is a reduced quadratic irrational. The same computation also gives the Lagrange constant of a non-reduced quadratic irrational. We explain this in this section.

We first show that the study of quadratic irrationals can be reduced to the study of reduced quadratic irrationals. The following proposition actually holds for arbitrary irrational numbers.

PROPOSITION 3.3.1. *Let α and β be $GL(2,\mathbb{Z})$-equivalent irrational numbers. Then*

$$\mathcal{L}(\alpha) = \mathcal{L}(\beta).$$

PROOF. Write

$$\alpha = [a_0; a_1, \dots], \qquad \beta = [b_0; b_1, \dots].$$

By Theorem 2.4.6, there exist $n, m \in \mathbb{Z}_{\geq 0}$ such that $a_{n+h} = b_{m+h}$ for every $h \in \mathbb{Z}_{\geq 0}$. Thus, after deleting finitely many initial partial quotients, the two continued-fraction expansions have the same tail. Write this common tail as $\gamma = [c_0; c_1, c_2, \dots]$.

For $j \geq 1$, put

$$\rho_j := [c_j; c_{j+1}, \dots] + [0; c_{j-1}, \dots, c_0, a_{n-1}, \dots, a_1],$$
$$\sigma_j := [c_j; c_{j+1}, \dots] + [0; c_{j-1}, \dots, c_0, b_{m-1}, \dots, b_1].$$

If $n = 0$ or $m = 0$, the corresponding finite list $a_{n-1}, \dots, a_1$ or $b_{m-1}, \dots, b_1$ is read as empty. Up to deleting finitely many initial terms, the sequence computing $\mathcal{L}(\alpha)$ is $(\rho_j)_{j\geq 1}$ and the sequence computing $\mathcal{L}(\beta)$ is $(\sigma_j)_{j\geq 1}$. Hence

$$\mathcal{L}(\alpha) = \limsup_{j\to\infty} \rho_j, \qquad \mathcal{L}(\beta) = \limsup_{j\to\infty} \sigma_j.$$

The first terms of ρ_j and σ_j are equal; the difference comes only from the second terms. These second terms have the common initial block $0, c_{j-1}, \dots, c_0$ of length $j+1$. Hence Lemma 3.1.7 implies, for $j \geq 2$,

$$|\rho_j - \sigma_j| \leq \frac{1}{j(j-1)}.$$

Thus $|\rho_j - \sigma_j| \to 0$. Since two real sequences whose difference tends to zero have the same limit superior, it follows that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$. ☐

COROLLARY 3.3.2. *Let α be a quadratic irrational with continued-fraction expansion*

$$\alpha = [a_0; a_1, \dots, a_k, \overline{c_1, \dots, c_n}].$$

Define reduced quadratic irrationals

$$\gamma_1 := [\overline{c_1, c_2, \dots, c_n}], \quad \gamma_2 := [\overline{c_2, c_3, \dots, c_n, c_1}], \quad \dots, \gamma_n := [\overline{c_n, c_1, \dots, c_{n-1}}].$$

Then

$$\mathcal{L}(\alpha) = \mathcal{L}(\gamma_1) = \mathcal{L}(\gamma_2) = \cdots = \mathcal{L}(\gamma_n).$$

PROOF. This follows from Theorem 2.4.6 and Proposition 3.3.1. ☐

The preceding argument shows that, in order to study the Lagrange constant of a quadratic irrational, it suffices to study an equivalent reduced quadratic irrational. The main theorem of this section is the following.

THEOREM 3.3.3. *Let $k \geq 1$ and let $b_0, \dots, b_{k-1} \in \mathbb{Z}_{\geq 1}$. Suppose that a reduced quadratic irrational α has infinite continued-fraction expansion*

$$\alpha = [\overline{b_0, b_1, \dots, b_{k-1}}].$$

Let $\mathbf{b}$ be the bi-infinite sequence obtained by repeating the period $b_0, b_1, \dots, b_{k-1}$ indefinitely in both directions:

$$\mathbf{b} = (\dots, b_0, b_1, \dots, b_{k-1}, b_0, b_1, \dots, b_{k-1}, b_0, b_1, \dots, b_{k-1}, \dots).$$

Then

$$\mathcal{L}(\alpha) = \mathcal{S}(\mathbf{b}).$$

PROOF. Write $\alpha = [a_0; a_1, \dots]$, with the indices now taken over the natural numbers. Since $(a_n)_{n=0}^{\infty}$ is bounded, we have $\mathcal{L}(\alpha) < \infty$. Put

$$\alpha_{i+1} = [a_{i+1}; a_{i+2}, \dots], \qquad \beta_i = [a_i; a_{i-1}, \dots].$$

To understand the accumulation points of $(\lambda_n(\alpha))_{n=1}^{\infty}$, first consider the accumulation points of $(\alpha_{n+1})_{n=1}^{\infty}$. By pure periodicity, this sequence is periodic when the index is read modulo k. Thus its accumulation points are precisely the k values

$$\theta_i := \alpha_i \qquad (0 \le i \le k-1).$$

Henceforth, read all indices modulo k, so that $b_j := b_{j'}$ whenever $0 \le j' \le k-1$ and $j \equiv j' \pmod{k}$. If $\alpha_{n+1} = \alpha_i$, then β_n is the finite continued fraction

$$\beta_n = [b_{i-1}; b_{i-2}, \dots, b_1, b_0, b_{k-1}, \dots, b_1, b_0, b_{k-1}, \dots, b_1],$$

where the cyclic block $b_{i-1}, b_{i-2}, \dots, b_i$ appears finitely many times and the last component is the one corresponding to the actual index 1. This occurs when $n+1 = mk+i$ for some $m \in \mathbb{Z}_{\ge 0}$, and then the periodic block occurs m times in the finite continued fraction for β_n. Therefore, if a subsequence $(\alpha_{n_j+1})_{j=1}^{\infty}$ converges to α_i, then the corresponding subsequence $(\beta_{n_j})_{j=1}^{\infty}$ has only one possible accumulation point, namely

$$\eta_i := [\overline{b_{i-1}, b_{i-2}, \dots, b_1, b_0, b_{k-1}, \dots, b_i}].$$

Thus all accumulation points of $(\lambda_n(\alpha))_{n=1}^{\infty}$ are

$$\left\{ r_i := \theta_i + \frac{1}{\eta_i} \;\middle|\; 0 \le i \le k-1 \right\}.$$

For each i, we have $r_i = \ell_i(\mathbf{b})$. Since $\mathcal{L}(\alpha)$ is the supremum of these accumulation points, and the set is finite, it is the maximum of the r_i. Hence

$$\mathcal{L}(\alpha) = \mathcal{S}(\mathbf{b}).$$

□

As a consequence we obtain the following statement.

COROLLARY 3.3.4. *Let α be a quadratic irrational with infinite continued-fraction expansion*

$$\alpha = [a_0; a_1, \dots, a_n, \overline{b_0, b_1, \dots, b_{k-1}}]$$

for some $n \ge 0$ and $k \ge 1$. Let $\mathbf{b}$ be the bi-infinite sequence obtained by repeating the period $b_0, b_1, \dots, b_{k-1}$ indefinitely in both directions:

$$\mathbf{b} = (\dots, b_0, b_1, \dots, b_{k-1}, b_0, b_1, \dots, b_{k-1}, b_0, b_1, \dots, b_{k-1}, \dots).$$

Then

$$\mathcal{L}(\alpha) = \mathcal{S}(\mathbf{b}).$$

In this corollary, the candidates for the supremum value of $\mathcal{S}(\mathbf{b})$ are finite in number, so $\mathcal{L}(\alpha)$ can in principle be computed by hand.

We end this chapter by expressing the Lagrange constant of α in terms of continued-fraction matrices associated with the continued-fraction expansion. This makes the computation of $\mathcal{L}(\alpha)$ still easier. For a finite sequence $(a_0, \dots, a_k)$, define

$$F_{(a_0, a_1, \dots, a_k)} := \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}.$$

THEOREM 3.3.5. *Let α be a quadratic irrational, and suppose that a nonempty period of its infinite continued-fraction expansion is $(b_0, \dots, b_{k-1})$, so $k \ge 1$. For $0 \le i \le k-1$, put*

$$S_i := (b_i, b_{i+1}, \dots, b_{k-1}, b_0, \dots, b_{i-1}).$$

Then

$$\mathcal{L}(\alpha) = \max\left\{ \frac{\sqrt{(\operatorname{tr}(F_{S_i}))^2 - (-1)^k \cdot 4}}{(F_{S_i})_{21}} \;\middle|\; 0 \le i \le k-1 \right\}.$$

Here $(F_{S_i})_{21}$ *denotes the* $(2,1)$*-entry of* F_{S_i}*.*

Proof. Read the indices modulo k. Put

$$\theta_i = [\overline{b_i, b_{i+1}, \ldots, b_{k-1}, b_0, \ldots, b_{i-1}}],$$

and

$$\eta_i = [\overline{b_{i-1}, b_{i-2}, \ldots, b_0, b_{k-1}, \ldots, b_i}].$$

By Corollary 3.3.4, it suffices to show

$$\theta_i + \frac{1}{\eta_i} = \frac{\sqrt{(\operatorname{tr}(F_{S_i}))^2 - (-1)^k \cdot 4}}{(F_{S_i})_{21}}.$$

Since

$$\theta_i = [b_i; b_{i+1}, \ldots, b_{i-1}, \theta_i],$$

Proposition 2.4.5 gives, where F_{S_i} acts by fractional linear transformations,

$$\theta_i = F_{S_i}\theta_i.$$

Write

$$F_{S_i} = \begin{bmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{bmatrix}.$$

From the definition of the action,

$$\theta_i = \frac{\theta_i p_{k-1} + p_{k-2}}{\theta_i q_{k-1} + q_{k-2}}.$$

Solving this quadratic equation gives

$$\begin{aligned}
\theta_i &= \frac{p_{k-1} - q_{k-2} + \sqrt{(p_{k-1} - q_{k-2})^2 + 4p_{k-2}q_{k-1}}}{2q_{k-1}} \\
&= \frac{p_{k-1} - q_{k-2} + \sqrt{(p_{k-1} + q_{k-2})^2 - 4(p_{k-1}q_{k-2} - p_{k-2}q_{k-1})}}{2q_{k-1}} \\
&= \frac{p_{k-1} - q_{k-2} + \sqrt{(\operatorname{tr}(F_{S_i}))^2 - (-1)^k \cdot 4}}{2(F_{S_i})_{21}}.
\end{aligned}$$

We take the plus sign in front of the square root because θ_i is larger than its quadratic conjugate θ_i'. On the other hand, by Proposition 2.5.10,

$$\eta_i = -\frac{1}{\theta_i'}.$$

Therefore

$$\frac{1}{\eta_i} = -\frac{p_{k-1} - q_{k-2} - \sqrt{(\operatorname{tr}(F_{S_i}))^2 - (-1)^k \cdot 4}}{2(F_{S_i})_{21}}.$$

Adding the two expressions gives the desired formula. □

In the proof of Theorem 3.3.5, the numerator in the expression for $\theta_i + 1/\eta_i$ depends on the trace of F_{S_i}. In fact this trace is independent of i. Indeed,

$$F_{S_{i+1}} = \begin{bmatrix} b_i & 1 \\ 1 & 0 \end{bmatrix}^{-1} F_{S_i} \begin{bmatrix} b_i & 1 \\ 1 & 0 \end{bmatrix},$$

where $S_k := S_0$. Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we have

$$\operatorname{tr}(F_{S_{i+1}}) = \operatorname{tr}\left(\begin{bmatrix} b_i & 1 \\ 1 & 0 \end{bmatrix}^{-1} F_{S_i} \begin{bmatrix} b_i & 1 \\ 1 & 0 \end{bmatrix} \right) = \operatorname{tr}\left(F_{S_i} \begin{bmatrix} b_i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_i & 1 \\ 1 & 0 \end{bmatrix}^{-1} \right) = \operatorname{tr}(F_{S_i}).$$

Thus, when computing the candidates for $\mathcal{L}(\alpha)$, the numerator only has to be computed once. Since this common numerator is positive, the maximum is attained precisely when the denominator $(F_{S_i})_{21}$ is minimal. We record this as a proposition.

Proposition 3.3.6. *In the setting of Theorem 3.3.5, let* $j \in \{0, \ldots, k-1\}$ *satisfy*

$$\min_{0 \le i \le k-1} \{(F_{S_i})_{21}\} = (F_{S_j})_{21}.$$

Then

$$\mathcal{L}(\alpha) = \frac{\sqrt{(\operatorname{tr}(F_{S_j}))^2 - (-1)^k \cdot 4}}{(F_{S_j})_{21}}.$$

Let us compute the Lagrange constant for a concrete quadratic irrational.

Example 3.3.7. We compute the Lagrange constant of $\alpha = 1 + \sqrt{3}$. First we find its infinite continued-fraction expansion. Since

$$\alpha - 2 = \sqrt{3} - 1 = \frac{2}{\sqrt{3}+1},$$

we have

$$\frac{1}{\alpha - 2} = \frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2} = 1 + \frac{1}{\alpha}.$$

Therefore

$$\alpha = 2 + \frac{1}{1 + \frac{1}{\alpha}}.$$

Hence $\alpha = [\overline{2, 1}]$, the period is $(2, 1)$, and its length is $k = 2$. We now compute $\mathcal{L}(\alpha)$ using Theorem 3.3.5.

The rotations of the period are

$$S_0 = (2, 1), \qquad S_1 = (1, 2).$$

The corresponding continued-fraction matrices are

$$F_{(2,1)} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \qquad F_{(1,2)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

In both cases $\operatorname{tr}(F_{S_i}) = 4$. On the other hand,

$$(F_{(2,1)})_{21} = 1, \qquad (F_{(1,2)})_{21} = 2.$$

Since $k = 2$, we have $(-1)^k = 1$. Hence Theorem 3.3.5 gives

$$\mathcal{L}(\alpha) = \max_{i=0,1} \left\{ \frac{\sqrt{(\operatorname{tr}(F_{S_i}))^2 - 4}}{(F_{S_i})_{21}} \right\} = \max \left\{ \frac{\sqrt{12}}{1}, \frac{\sqrt{12}}{2} \right\} = 2\sqrt{3}.$$

Thus $\mathcal{L}(1 + \sqrt{3}) = 2\sqrt{3}$.

The preceding discussion shows that, for the Lagrange spectrum of quadratic irrationals, once we know which cyclic cut of the period makes $(F_{S_j})_{21}$ minimal, the value can be computed. At present, however, the most direct way to determine this cut is simply to compute all the candidates and compare them. For certain classes of quadratic irrationals there are methods that avoid this brute-force comparison, and these classes will be studied more deeply in Part II.

CHAPTER 4

Markov Spectrum

In this chapter we discuss the Markov spectrum. The Markov spectrum arises from a minimum problem for indefinite binary quadratic forms, and at first sight it may look quite different from the Lagrange spectrum studied in the preceding chapter. Once both spectra are rewritten in terms of continued fractions and bi-infinite sequences, however, their structures become very similar.

We first define the Markov constant attached to a binary quadratic form and examine its meaning through concrete examples. Next, using canonical reduced binary quadratic forms and the action of the unimodular group, we choose representatives in each class of quadratic forms for which the Markov constant is easier to compute. Then, in analogy with the Lagrange constant from the preceding chapter, we express the Markov constant by means of a bi-infinite sequence, so that the two spectra can be compared within a common framework. Finally, we show that the Markov constant of a binary quadratic form with rational coefficients agrees with the Lagrange constant of the corresponding quadratic irrational. Thus three objects give the same value: the Lagrange constant of a quadratic irrational, the Markov constant of a rational binary quadratic form, and the value of $\mathcal{S}$ obtained from a periodic bi-infinite sequence.

This chapter follows mainly [**Reu19, LMMR20**].

1. Definitions and First Examples

We begin by introducing the set called the Markov spectrum.

DEFINITION 4.1.1. Let Q be a real binary quadratic form. We assume that Q is indefinite; namely, if $Q(x,y) = ax^2+bxy+cy^2$, then $D(Q) := b^2-4ac > 0$. We also assume that $Q(x,y) \neq 0$ for every lattice point $(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then

$$\mathcal{M}(Q) := \frac{\sqrt{D(Q)}}{\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(x,y)|}$$

is called the *Markov constant* attached to Q. If the infimum in the denominator is 0, we put $\mathcal{M}(Q) = \infty$. The set of all Markov constants

$$\mathcal{M} := \left\{ \mathcal{M}(Q) \;\middle|\; \begin{array}{l} Q(x,y) = ax^2+bxy+cy^2,\ a,b,c\in\mathbb{R},\ D = b^2-4ac > 0, \\ Q(x,y) \neq 0 \text{ for every } (x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\} \end{array} \right\}$$

is called the *Markov spectrum*.

In what follows, all quadratic forms are real binary quadratic forms, so we will simply call them quadratic forms. Let us examine more carefully the condition under which the Markov constant of $Q(x,y) = ax^2+bxy+cy^2$ is defined in the sense used in this text. Here this means that Q has no nonzero lattice zero; the possibility that the denominator is 0, and hence that $\mathcal{M}(Q) = \infty$, is treated separately.[1] We first record the following proposition.

PROPOSITION 4.1.2. *Let $Q(x,y) = ax^2+bxy+cy^2$, where $a,b,c \in \mathbb{R}$, $D > 0$, and $a \neq 0$. Then the following two conditions are equivalent:*

(1) *there is no lattice point $(\alpha,\beta) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $Q(\alpha,\beta) = 0$;*
(2) *the polynomial $Q(x,1)$ has two distinct irrational roots.*

[1] In this text, the assertion that $\mathcal{M}(Q) = \infty$ is distinguished from the assertion that $\mathcal{M}(Q)$ is not well defined.

Proof. Since $D(Q) > 0$, the polynomial $Q(x,1)$ has no multiple root. Suppose that a nonzero lattice point (α,β) satisfies $Q(\alpha,\beta) = 0$. If $\beta = 0$, then $a\alpha^2 = 0$, contradicting $a \neq 0$ and $\alpha \neq 0$. Hence $\beta \neq 0$. Dividing $Q(\alpha,\beta) = 0$ by β^2, we obtain

$$a\left(\frac{\alpha}{\beta}\right)^2 + b\left(\frac{\alpha}{\beta}\right) + c = 0.$$

Thus $x = \alpha/\beta \in \mathbb{Q}$ is a root of $Q(x,1) = 0$. Therefore at least one root of $Q(x,1)$ is rational.

Conversely, suppose that one of the two roots of $Q(x,1)$ is rational. Then $Q(x,1) = 0$ has a rational solution $x = \alpha/\beta$, written in lowest terms. Multiplying by β^2, we obtain $Q(\alpha,\beta) = 0$. □

We will also use the following elementary observation.

Proposition 4.1.3. *If $\mathcal{M}(Q)$ is well defined, then $a \neq 0$ and $c \neq 0$.*

Proof. If $a = 0$, then $Q(1,0) = 0$, so $\mathcal{M}(Q)$ is not defined. If $c = 0$, then $Q(0,1) = 0$, and again $\mathcal{M}(Q)$ is not defined. Hence both a and c must be nonzero. □

The preceding two propositions give the following corollary.

Corollary 4.1.4. *The constant $\mathcal{M}(Q)$ is well defined if and only if, for $Q(x,y) = ax^2 + bxy + cy^2$, the polynomial $Q(x,1)$ has two irrational roots.*

Proof. If $\mathcal{M}(Q)$ is well defined, then $a \neq 0$ by Proposition 4.1.3, and Q has no nonzero lattice zero. Hence Proposition 4.1.2 shows that $Q(x,1)$ has two irrational roots. Conversely, if $Q(x,1)$ has two irrational roots, then $a \neq 0$, and Proposition 4.1.2 shows that Q has no nonzero lattice zero. Thus $\mathcal{M}(Q)$ is well defined. □

Example 4.1.5. As in the case of the Lagrange spectrum, let us look at concrete examples.

(1) Let $Q(x,y) = x^2 - xy - y^2$. Then $D(Q) = 5$, and the roots of $Q(x,1)$ are $(1 \pm \sqrt{5})/2$. Hence $Q(x,y) \neq 0$ for every nonzero lattice point. Since Q has integer coefficients, $|Q(\alpha,\beta)| \in \mathbb{Z}_{\geq 1}$ for every lattice point (α,β) with $Q(\alpha,\beta) \neq 0$. Moreover $Q(1,0) = 1$, so

$$\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(x,y)| = 1.$$

Therefore

$$\mathcal{M}(Q) = \frac{\sqrt{5}}{1} = \sqrt{5}.$$

(2) Let $Q(x,y) = x^2 - 2xy - y^2$. Then $D(Q) = 8$, and the roots of $Q(x,1)$ are $1 \pm \sqrt{2}$. Hence $Q(x,y) \neq 0$ for every nonzero lattice point. Again Q has integer coefficients, so $|Q(\alpha,\beta)| \in \mathbb{Z}_{\geq 1}$ for every lattice point (α,β) with $Q(\alpha,\beta) \neq 0$. Since $Q(1,0) = 1$, we have

$$\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(x,y)| = 1.$$

Thus

$$\mathcal{M}(Q) = \frac{\sqrt{8}}{1} = 2\sqrt{2}.$$

2. Unimodular Group Orbits of Binary Quadratic Forms

In this section, as preparation for computing Markov constants, we decompose quadratic forms into orbits under the unimodular group. We will also see that in each orbit one may choose a representative with good properties, called a canonical reduced quadratic form. In Chapter 2, Section 5, we carried out an analogous discussion for quadratic irrationals; the present discussion can be viewed as the counterpart for quadratic forms.

First we introduce the unimodular group action on the set of quadratic forms for which $\mathcal{M}(Q)$ is well defined. Put

$$\mathcal{Q} := \left\{ Q\colon \mathbb{R}^2 \to \mathbb{R} \;\middle|\; \begin{array}{l} Q(x,y) = ax^2 + bxy + cy^2,\ a,b,c \in \mathbb{R},\ D(Q) = b^2 - 4ac > 0, \\ Q(x,y) \neq 0 \text{ for every } (x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\} \end{array} \right\}.$$

By Corollary 4.1.4, $Q(x,1)$ always has two irrational roots for $Q \in \mathcal{Q}$. For $A = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in GL(2,\mathbb{Z})$ and $Q(x,y) = ax^2 + bxy + cy^2 \in \mathcal{Q}$, define QA by

$$QA(x,y) := Q(px + qy, rx + sy). \tag{4.2.1}$$

We have the following.

THEOREM 4.2.1. *For $A \in GL(2,\mathbb{Z})$ and $Q \in \mathcal{Q}$, one has $QA \in \mathcal{Q}$. Moreover, this operation gives a right action $\mathcal{Q} \curvearrowleft GL(2,\mathbb{Z})$.*

PROOF. We first show that $QA \in \mathcal{Q}$ for $Q \in \mathcal{Q}$. We have

$$\begin{aligned} QA(x,y) &= Q(px + qy, rx + sy) \\ &= (ap^2 + bpr + cr^2)x^2 + (2apq + b(ps + qr) + 2crs)xy + (aq^2 + bqs + cs^2)y^2. \end{aligned}$$

Thus

$$D(QA) = (2apq + b(ps+qr) + 2crs)^2 - 4(ap^2 + bpr + cr^2)(aq^2 + bqs + cs^2) = (b^2 - 4ac)(ps - qr)^2.$$

Since $ps - qr = \pm 1$, we obtain $D(QA) = D(Q)$, and in particular $D(QA) > 0$. Next regard A as the linear transformation of $\mathbb{R}^2$ given by

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} px + qy \\ rx + sy \end{bmatrix}.$$

Since $A \in GL(2,\mathbb{Z})$, this restricts to a bijection

$$\mathbb{Z}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{Z}^2 \setminus \{(0,0)\}.$$

Hence, because Q has no nonzero lattice zero, the same is true of QA. Therefore $QA \in \mathcal{Q}$. Finally, since E_2 denotes the 2×2 identity matrix, we have $QE_2 = Q$, and $(QB)A = Q(BA)$ follows immediately from viewing A and B as linear transformations of $\mathbb{R}^2$. □

We now introduce unimodular equivalence on $\mathcal{Q}$.

DEFINITION 4.2.2. Let $Q, R \in \mathcal{Q}$. If there exists $A \in GL(2,\mathbb{Z})$ such that $R = QA$, then Q and R are called *unimodularly equivalent*; in what follows we simply say *equivalent*. We write $Q \sim R$. The equivalence class

$$O_Q = \{R \mid R \sim Q\}$$

is called the *unimodular orbit* of Q, or simply the *orbit* of Q.

Let $\mathcal{Q}(d)$ denote the set of all elements of $\mathcal{Q}$ with $D(Q) = d$. In the proof of Theorem 4.2.1, we saw that the unimodular group action preserves the discriminant. Hence we have the following corollary.

COROLLARY 4.2.3. *The action (4.2.1) of the unimodular group on $\mathcal{Q}$ restricts to a right action on $\mathcal{Q}(d)$.*

In Corollary 2.5.6 of Chapter 2, Section 5, we saw that a quadratic irrational α is unimodularly equivalent to a reduced quadratic irrational β. A similar statement holds for elements of $\mathcal{Q}$. To state it, we first define reduced quadratic forms.

DEFINITION 4.2.4. Let $Q(x,y) = ax^2 + bxy + cy^2$ be an indefinite quadratic form. Suppose that $Q(x,1)$ has two distinct roots α, β satisfying

$$|\alpha| > 1, \qquad |\beta| < 1, \qquad \alpha\beta < 0.$$

Then Q is called a *reduced quadratic form*. If, in addition, the root with absolute value greater than 1 satisfies $\alpha > 1$, then Q is called a *canonical reduced quadratic form*.[2]

[2] This terminology is not universal; in many texts such a form is simply called a reduced quadratic form.

It may seem asymmetric to define reducedness by looking at the roots of $Q(x,1)$, since this appears to distinguish x and y. In fact no symmetry is lost. The roots of $Q(x,1)$ and those of $Q(1,y)$ are reciprocal to each other, as long as the roots are nonzero. Therefore one obtains an equivalent definition by imposing the corresponding condition on the roots of $Q(1,y)$.

When the coefficients a,b,c of $Q(x,y) = ax^2 + bxy + cy^2$ are rational, if one root of $Q(x,1)$ is irrational, then the other root is also irrational, and the two roots are quadratic conjugates. If Q is a canonical reduced quadratic form, then its roots satisfy $\alpha > 1$ and $-1 < \beta < 0$, so α is a reduced quadratic irrational. Conversely, if α is a reduced quadratic irrational, then any rational-coefficient quadratic form having α as a root is a canonical reduced quadratic form. From this viewpoint, canonical reduced quadratic forms may be regarded as a generalization of reduced quadratic irrationals to the setting of arbitrary irrational roots.

Let $\mathcal{R}$ be the set of all canonical reduced quadratic forms in $\mathcal{Q}$, and let $\mathcal{R}(d)$ be the subset consisting of those with discriminant d. We prove the following theorem.

THEOREM 4.2.5. *For every $Q \in \mathcal{Q}(d)$, there exists $R \in \mathcal{R}(d)$ such that $Q \sim R$. Hence every orbit in $\mathcal{Q}(d)$ has a representative in $\mathcal{R}(d)$. In particular, every orbit in $\mathcal{Q}$ has a representative in $\mathcal{R}$.*

We first prove a lemma.

LEMMA 4.2.6. *Let $A = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in GL(2,\mathbb{Z})$ and let $Q \in \mathcal{Q}$. If the roots of $Q(x,1)$ are α and β, then the roots of $QA^{-1}(x,1)$ are*

$$\frac{p\alpha + q}{r\alpha + s}, \qquad \frac{p\beta + q}{r\beta + s}.$$

Similarly, if the roots of $Q(1,y)$ are α and β, then the roots of $QA^{-1}(1,y)$ are

$$\frac{s\alpha + r}{q\alpha + p}, \qquad \frac{s\beta + r}{q\beta + p}.$$

PROOF. We prove the first assertion. Since the roots α, β are irrational and p,q,r,s are integers, the denominators $r\alpha + s$ and $r\beta + s$ do not vanish. Since $\det A = \pm 1$, the inverse of A is either $\left[\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right]$ or its negative. Because a quadratic form is homogeneous of degree two, this overall sign does not affect the value of the form. Hence, for the purpose of finding the roots, we may compute

$$QA^{-1}(x,1) = Q(sx - q, -rx + p) = (-rx + p)^2 Q\left(\frac{sx - q}{-rx + p}, 1\right).$$

Since $Q(\alpha, 1) = 0$, the equality

$$\alpha = \frac{sx - q}{-rx + p}$$

implies that $QA^{-1}(x,1) = 0$. Solving this equality for x gives

$$x = \frac{p\alpha + q}{r\alpha + s}.$$

Thus this is a root of $QA^{-1}(x,1)$. The argument for β is identical. The second assertion is proved in the same way, using the roots of $Q(1,y)$ instead of those of $Q(x,1)$. ☐

PROOF OF THEOREM 4.2.5. Let α, β be the roots of $Q(x,1)$, ordered so that $\alpha > \beta$. By Proposition 4.1.2, both roots are irrational. If the roots already satisfy $\alpha > 0 > \beta$, we do nothing at this stage. If $0 > \alpha > \beta$, choose a sufficiently large integer h and replace Q by $Q\left[\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right]^{-1}$. By Lemma 4.2.6, the new roots are $\alpha + h$ and $\beta + h$, so we are reduced to the case where both roots are positive.

It remains to handle the case $\alpha > \beta > 0$. During the process of deleting common initial partial quotients, the order of the two roots may be reversed. Whenever this happens, we rename the larger root α and the smaller root β. Write the infinite regular continued-fraction expansions as

$$\alpha = [a_0; a_1, \ldots], \qquad \beta = [b_0; b_1, \ldots].$$

Let m be the smallest index such that $a_m \neq b_m$. We construct a quadratic form Q' equivalent to Q whose roots α', β' satisfy $\alpha' > 0 > \beta'$. If $m = 0$, then after interchanging the names of the roots if necessary, we may assume $a_0 > b_0$. Then $\alpha - a_0 > 0 > \beta - a_0$. By Lemma 4.2.6, a form whose roots are $\alpha - a_0$ and $\beta - a_0$ is obtained by taking $Q' = Q\left[\begin{smallmatrix} 1 & -a_0 \\ 0 & 1 \end{smallmatrix}\right]^{-1}$. This gives the desired Q'. If $m \neq 0$, put $Q_1 := Q\left[\begin{smallmatrix} 0 & 1 \\ 1 & -a_0 \end{smallmatrix}\right]^{-1}$. Then the roots of $Q_1(x, 1)$ are

$$\alpha_1 = [a_1; a_2, \dots], \qquad \beta_1 = [b_1; b_2, \dots].$$

Thus replacing Q by the equivalent form Q_1 decreases the value of m by at least one. Repeating this operation until $m = 0$, and then applying the argument above, we obtain a form Q' whose two roots have opposite signs.

Therefore, after replacing Q by an equivalent form if necessary, we may assume that the roots of $Q(x, 1)$ are α, β with $\alpha > 0 > \beta$. If $|\alpha| > 1$ and $|\beta| < 1$, then $R = Q$ is already the desired form. If $|\alpha| > 1$ and $|\beta| > 1$, choose an integer h such that $-1 < \beta + h < 0$ and take $R = Q\left[\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right]^{-1}$. If $|\alpha| < 1$ and $|\beta| < 1$, take $\widetilde{Q} := Q\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]^{-1}$. The roots of $\widetilde{Q}(x, 1)$ are $1/\alpha$ and $1/\beta$, so this case is reduced to one of the preceding cases. Finally, if $|\alpha| < 1$ and $|\beta| > 1$, then $R = \widetilde{Q}$ is canonical reduced. This proves the theorem. □

Lemma 4.2.6 says that the unimodular action on quadratic forms induces the usual unimodular action on their roots. It therefore allows us to reinterpret part of Chapter 2, Section 5. For example, Corollary 2.5.6 can be proved immediately by applying Theorem 4.2.5 to a form Q such that $Q(x, 1)$ has α as a root. We should note, however, that Theorem 2.5.5, which underlies Corollary 2.5.6 in Chapter 2, Section 5, uses special properties of quadratic irrationals; it cannot be extended to arbitrary quadratic forms. This perspective nevertheless shows that the theory of quadratic forms developed here is a natural generalization of the corresponding theory for quadratic irrationals.

3. A Bi-infinite Sequence Formula for the Markov Constant

We now explain how to compute the Markov constant by means of a bi-infinite sequence, in parallel with the Lagrange constant. The next corollary is a refinement of Theorem 4.2.1 and follows directly from it.

COROLLARY 4.3.1. *If $Q, R \in \mathcal{Q}$ and $Q \sim R$, then $\mathcal{M}(Q) = \mathcal{M}(R)$.*

PROOF. By definition,

$$\mathcal{M}(Q) = \frac{\sqrt{D(Q)}}{\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(x, y)|}.$$

It suffices to show that $D(Q) = D(R)$ and that the two infima in the denominators are equal. The equality $D(Q) = D(R)$ was proved in the proof of Theorem 4.2.1. Moreover, an element $A \in GL(2, \mathbb{Z})$ acts on $\mathbb{R}^2$ as a bijection that sends lattice points to lattice points and the origin to the origin. Hence

$$\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |R(x, y)| = \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(px + qy, rx + sy)| = \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q(x, y)|.$$

The claim follows. □

COROLLARY 4.3.2. *The following equality holds:*

$$\mathcal{M} = \{\mathcal{M}(Q) \mid Q \in \mathcal{R}\}.$$

PROOF. This follows immediately from Theorem 4.2.5 and Corollary 4.3.1. □

The next theorem is the main result of this section.

THEOREM 4.3.3. *Let $Q \in \mathcal{R}$, and let the two irrational roots of $Q(x, 1)$ be θ and $-\frac{1}{\eta}$, where $\theta, \eta > 1$. From the continued-fraction expansions*

$$\theta = [a_0; a_1, \dots], \qquad \eta = [a_{-1}; a_{-2}, \dots],$$

form the bi-infinite sequence

$$\mathbf{a} = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots).$$

Then

$$\mathcal{M}(Q) = \mathcal{S}(\mathbf{a}).$$

Before proving the theorem, let us spell out what it means. For the Lagrange constant, one could construct a bi-infinite continued fraction from an infinite continued-fraction expansion only in the quadratic irrational case. For the Markov constant the situation is simpler: as long as $Q \in \mathcal{R}$, the infinite continued-fraction expansions of the irrational roots of $Q(x, 1)$ always produce a bi-infinite sequence. Conversely, by choosing arbitrary irrational numbers $\theta, \eta > 1$ and setting

$$Q = (x - \theta y)\left(x + \frac{1}{\eta} y\right),$$

one obtains an element $Q \in \mathcal{R}$ corresponding to any given bi-infinite sequence $\mathbf{a}$ of positive integers. We therefore get the following corollary.

Corollary 4.3.4. *One has $\mathcal{M} = \mathcal{S}$. In particular, Theorem 3.2.5 implies $\mathcal{L} \subset \mathcal{M}$.*

We now prepare the proof of Theorem 4.3.3. For $\delta \in \mathbb{Z}$, put $F_\delta := \left[\begin{smallmatrix} \delta & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Lemma 4.3.5. *Let $Q \in \mathcal{R}$, and suppose that the roots of $Q(x, 1)$ are θ and $-1/\eta$, where $\theta, \eta > 1$. Then QF_δ^{-1} and QF_ε belong to $\mathcal{R}$ if and only if, respectively,*

$$\delta = \delta_0 := \lfloor \eta \rfloor, \qquad \varepsilon = \varepsilon_0 := \lfloor \theta \rfloor.$$

We call $QF_{\delta_0}^{-1}$ and QF_{ε_0} the *left neighbor* and the *right neighbor* of Q, respectively.

Proof. Take $Q(x, y) = ax^2 + bxy + cy^2 \in \mathcal{R}$. Since $F_\gamma \in GL(2, \mathbb{Z})$, both QF_γ and QF_γ^{-1} belong to $\mathcal{Q}$ for every $\gamma \in \mathbb{Z}$. Thus it remains to determine exactly when these forms are canonical reduced.

First consider QF_δ^{-1}. By Lemma 4.2.6, the roots of $QF_\delta^{-1}(x, 1)$ are

$$\delta + \frac{1}{\theta}, \qquad \delta - \eta.$$

The first number is always greater than 1 if $\delta \geq 1$. The condition that the second root lie in $(-1, 0)$ is

$$-1 < \delta - \eta < 0,$$

and this is equivalent to

$$\delta = \lfloor \eta \rfloor.$$

Thus QF_δ^{-1} is canonical reduced if and only if $\delta = \delta_0 := \lfloor \eta \rfloor$.

Next consider QF_ε. To apply Lemma 4.2.6, write $QF_\varepsilon = Q(F_\varepsilon^{-1})^{-1}$, where $F_\varepsilon^{-1} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -\varepsilon \end{smallmatrix}\right]$. The roots of $QF_\varepsilon(x, 1)$ are therefore

$$\frac{1}{\theta - \varepsilon}, \qquad \frac{1}{-\frac{1}{\eta} - \varepsilon} = -\frac{1}{\varepsilon + \frac{1}{\eta}}.$$

For QF_ε to be canonical reduced, the positive root must be greater than 1. This is equivalent to

$$0 < \theta - \varepsilon < 1,$$

and hence to

$$\varepsilon = \lfloor \theta \rfloor.$$

For this value of ε, the negative root is

$$-\frac{1}{\varepsilon + \frac{1}{\eta}},$$

which lies in $(-1, 0)$ because $\varepsilon = \lfloor \theta \rfloor \geq 1$. Thus QF_ε is canonical reduced if and only if $\varepsilon = \varepsilon_0 := \lfloor \theta \rfloor$. □

There is one point about the relation between right and left neighbors that must be checked. Namely, the right neighbor of the left neighbor of Q should return to Q, and conversely. This sounds obvious, since the matrices used to move to the right and to the left are inverse to each other. However, the integers δ and ε defining the left and right neighbors are determined separately for each quadratic form. Thus we must check that the value of δ for Q agrees with the value of ε for the left neighbor of Q.

PROPOSITION 4.3.6. *Let $Q_0 \in \mathcal{R}$. If Q_1 is the right neighbor of Q_0, then the left neighbor of Q_1 is Q_0. If Q_{-1} is the left neighbor of Q_0, then the right neighbor of Q_{-1} is Q_0.*

PROOF. Let the two roots of $Q_0(x,1)$ be θ_0 and $-1/\eta_0$, where $\theta_0, \eta_0 > 1$, and write

$$\theta_0 = [a_0; a_1, \dots], \qquad \eta_0 = [a_{-1}; a_{-2}, \dots].$$

For $i = \pm 1$, let the two roots of $Q_i(x,1)$ be θ_i and $-1/\eta_i$, where $\theta_i, \eta_i > 1$. We prove

$$Q_0 = Q_0 F_{\delta_0}^{-1} F_{\varepsilon_{-1}}, \qquad Q_0 = Q_0 F_{\varepsilon_0} F_{\delta_1}^{-1},$$

where

$$\delta_i := \lfloor \eta_i \rfloor, \qquad \varepsilon_i := \lfloor \theta_i \rfloor.$$

By definition, $\delta_0 = a_{-1}$ and $\varepsilon_0 = a_0$. If $Q_{-1} = Q_0 F_{\delta_0}^{-1}$, then Lemma 4.3.5 gives $\delta_0 = \lfloor \eta_0 \rfloor$, and hence

$$\theta_{-1} = \lfloor \eta_0 \rfloor + \frac{1}{\theta_0} = [a_{-1}; a_0, \dots], \qquad -\frac{1}{\eta_{-1}} = \lfloor \eta_0 \rfloor - \eta_0 = -[0; a_{-2}, a_{-3}, \dots].$$

Thus $\varepsilon_{-1} = a_{-1} = \delta_0$, and therefore

$$Q_0 = Q_0 F_{\delta_0}^{-1} F_{\varepsilon_{-1}}.$$

Next, if $Q_1 = Q_0 F_{\varepsilon_0}$, then Lemma 4.3.5 gives $\varepsilon_0 = \lfloor \theta_0 \rfloor$, and so

$$\theta_1 = \frac{1}{\theta_0 - \lfloor \theta_0 \rfloor} = \frac{1}{[0; a_1, a_2, \dots]} = [a_1; a_2, \dots], \qquad -\frac{1}{\eta_1} = \frac{1}{-\frac{1}{\eta_0} - \lfloor \theta_0 \rfloor} = -[0; a_0, a_{-1}, \dots].$$

Therefore $\eta_1 = [a_0; a_{-1}, \dots]$, and hence $\delta_1 = a_0 = \varepsilon_0$. It follows that

$$Q_1 F_{\delta_1}^{-1} = Q_0 F_{\varepsilon_0} F_{\delta_1}^{-1} = Q_0.$$

□

The preceding proposition makes the following definition well defined.

DEFINITION 4.3.7. Let $Q_0 \in \mathcal{R}$. Define a bi-infinite sequence of quadratic forms

$$(\dots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \dots)$$

inductively by requiring that Q_{i+1} be the unique right neighbor of Q_i and that Q_{i-1} be the unique left neighbor of Q_i. We call this sequence the *chain* of Q_0.

The next corollary describes the roots of the forms appearing in the chain of Q_0.

COROLLARY 4.3.8. *Let $Q_0 \in \mathcal{R}$, and consider the chain*

$$(\dots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \dots)$$

of Q_0. Suppose that the two roots of $Q_0(x,1)$ are θ_0 and $-1/\eta_0$, where $\theta_0, \eta_0 > 1$, and write

$$\theta_0 = [a_0; a_1, \dots], \qquad \eta_0 = [a_{-1}; a_{-2}, \dots].$$

For $i \in \mathbb{Z}$, let the two roots of $Q_i(x,1)$ be θ_i and $-1/\eta_i$, where $\theta_i, \eta_i > 1$. Then

$$\theta_i = [a_i; a_{i+1}, \dots], \qquad \eta_i = [a_{i-1}; a_{i-2}, \dots].$$

PROOF. The case $i = \pm 1$ is contained in the proof of Proposition 4.3.6. The general case follows by induction using the same argument. □

In the next proposition, the defining expression of the Markov constant appears naturally inside continued fraction theory. This makes the connection with continued fractions, and the strategy of the proof, more transparent.

PROPOSITION 4.3.9. *Let $Q_0 \in \mathcal{R}$, let*

$$(\ldots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \ldots)$$

be the chain of Q_0, and let $\theta_i, -1/\eta_i$ be the roots of $Q_i(x,1)$, with $\theta_i, \eta_i > 1$. If

$$\theta_0 = [a_0; a_1, \ldots], \qquad \eta_0 = [a_{-1}; a_{-2}, \ldots],$$

then

$$[a_i; a_{i+1}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots] = \theta_i + \frac{1}{\eta_i} = \frac{\sqrt{D(Q_0)}}{|Q_i(1,0)|}.$$

PROOF. The first equality follows immediately from Corollary 4.3.8. We prove the second one. Write

$$Q_i(x,y) = ax^2 + bxy + cy^2.$$

The numbers θ_i and $-1/\eta_i$ are the roots of $Q_i(x,1) = ax^2 + bx + c$, and $\theta_i > -1/\eta_i$. Thus, if $a > 0$, then

$$\theta_i = \frac{-b + \sqrt{D(Q_i)}}{2a}, \qquad -\frac{1}{\eta_i} = \frac{-b - \sqrt{D(Q_i)}}{2a},$$

whereas if $a < 0$, then

$$\theta_i = \frac{-b - \sqrt{D(Q_i)}}{2a}, \qquad -\frac{1}{\eta_i} = \frac{-b + \sqrt{D(Q_i)}}{2a}.$$

In the first case,

$$\theta_i + \frac{1}{\eta_i} = \frac{\sqrt{D(Q_i)}}{a} = \frac{\sqrt{D(Q_i)}}{|a|},$$

and in the second case,

$$\theta_i + \frac{1}{\eta_i} = -\frac{\sqrt{D(Q_i)}}{a} = \frac{\sqrt{D(Q_i)}}{|a|}.$$

Finally, $a = Q_i(1,0)$, and Q_i is equivalent to Q_0, so $D(Q_i) = D(Q_0)$. The result follows. $\square$

The next proposition is the final preparation for Theorem 4.3.3. Its proof is the most delicate part of this section.

PROPOSITION 4.3.10. *Let $Q_0 \in \mathcal{R}$, and let*

$$(\ldots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \ldots)$$

be the chain of Q_0. Then

$$\inf_{h\in\mathbb{Z}} |Q_h(1,0)| = \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_0(x,y)|. \tag{4.3.1}$$

PROOF. For every $h \in \mathbb{Z}$, the forms Q_h and Q_0 are equivalent. Hence

$$\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_h(x,y)| = \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_0(x,y)|.$$

By the definition of an infimum,

$$|Q_h(1,0)| \geq \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_h(x,y)|.$$

Therefore

$$\inf_{h\in\mathbb{Z}} |Q_h(1,0)| \geq \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_0(x,y)|.$$

It remains to prove

$$\inf_{h\in\mathbb{Z}} |Q_h(1,0)| \leq \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_0(x,y)|. \tag{4.3.2}$$

First suppose that the bi-infinite sequence

$$(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

determined by the two irrational roots θ_0 and $-1/\eta_0$ of $Q_0(x,1)$ satisfies $a_i = 1$ for all $i \in \mathbb{Z}$. Then, by Example 3.1.5 (1), we have $\theta_0 = \eta_0 = (1+\sqrt{5})/2$. Every quadratic form having θ_0 and $-1/\eta_0$ as roots is of the form

$$Q(x,y) = \lambda(x - \theta_0 y)\left(x + \frac{1}{\eta_0}y\right)$$

for some nonzero real number λ. Replacing Q by $\lambda^{-1}Q$ multiplies both sides of the desired equality of infima by $|\lambda|^{-1}$, and hence does not affect whether the equality holds. Thus we may assume $\lambda = 1$ and $Q_0(x,y) = x^2 - xy - y^2$. By Example 4.1.5 (1),

$$\inf_{h\in\mathbb{Z}} |Q_h(1,0)| = |Q_0(1,0)| = 1 = \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |Q_0(x,y)|.$$

In particular, (4.3.2) holds in this case.

Now suppose that $a_i \geq 2$ for some $i \in \mathbb{Z}$. By moving from Q_0 to a suitable left or right neighbor and then renaming it Q_0, we may assume that $a_0 \geq 2$. Then $\theta_0 > 2$. It suffices to show that for every $(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ there exists an index i such that

$$|Q_0(x,y)| \geq |Q_i(1,0)|.$$

If $|Q_0(x,y)| \geq |Q_0(1,0)|$, then we take $i = 0$. Hence assume $|Q_0(x,y)| < |Q_0(1,0)|$. Under this assumption, $y = 0$ is impossible, since

$$|Q_0(x,0)| = x^2|Q_0(1,0)| \geq |Q_0(1,0)|.$$

If $x = 0$, then, because Q_0 is the right neighbor of Q_{-1},

$$\begin{aligned}
|Q_0(0,y)| &= |Q_{-1}F_{\varepsilon_{-1}}(0,y)| \\
&= \left|Q_{-1}\begin{bmatrix} a_{-1} & 1 \\ 1 & 0 \end{bmatrix}(0,y)\right| \\
&= |Q_{-1}(y,0)| = y^2\,|Q_{-1}(1,0)| \geq |Q_{-1}(1,0)|\,.
\end{aligned}$$

Thus the case $x = 0$ is settled. Henceforth assume $x \neq 0$ and $y \neq 0$. Write

$$Q_0(x,y) = ay^2\left(\frac{x}{y} - \theta_0\right)\left(\frac{x}{y} + \frac{1}{\eta_0}\right). \tag{4.3.3}$$

First consider the case $x/y > 0$. Since $Q(x,y) = Q(-x,-y)$, we may assume $x, y > 0$. We claim that in fact $x/y \geq 2$. Suppose, to the contrary, that $x/y < 2$. Since $\theta_0 > 2$, we have $x/y < \theta_0$. Hence

$$|Q_0(x,y)| = |a|y^2\left|\frac{x}{y} - \theta_0\right|\left|\frac{x}{y} + \frac{1}{\eta_0}\right| > |a|y^2\left(2 - \frac{x}{y}\right)\left|\frac{x}{y}\right| = |a|(2y-x)x \geq |a|.$$

This contradicts $|Q_0(x,y)| < |Q_0(1,0)| = |a|$. Thus $x/y \geq 2$. Comparing the assumption $|Q_0(x,y)| < |Q_0(1,0)| = |a|$ with (4.3.3), we obtain

$$2y^2\left|\frac{x}{y} - \theta_0\right| < y^2\left|\frac{x}{y} - \theta_0\right|\left(\frac{x}{y} + \frac{1}{\eta_0}\right) < 1.$$

Therefore

$$\left|\theta_0 - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Write $x/y = p/q$ in lowest terms. Then $q \leq y$, and the above inequality gives

$$\left|\theta_0 - \frac{p}{q}\right| < \frac{1}{2y^2} \leq \frac{1}{2q^2}.$$

By Theorem 2.3.10, p/q is a convergent of θ_0. Thus there exists $n \geq 0$ such that

$$\frac{p}{q} = [a_0; a_1, \ldots, a_n] =: \frac{p_n}{q_n}.$$

Then

$$\begin{bmatrix} p_n \\ q_n \end{bmatrix} = F_{a_0}F_{a_1}\cdots F_{a_n}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Applying Q_0 to both sides gives

$$Q_0(p_n, q_n) = Q_{n+1}(1, 0).$$

Since p_n/q_n is in lowest terms, there exists $d \in \mathbb{Z}_{\geq 1}$ such that $(x, y) = (dp_n, dq_n)$. Therefore

$$|Q_0(x, y)| = d^2|Q_0(p_n, q_n)| \geq |Q_0(p_n, q_n)| = |Q_{n+1}(1, 0)|.$$

This proves the case $x/y > 0$.

Now assume $x/y < 0$. Replacing (x, y) by $(-x, -y)$ if necessary, we may assume $y > 0$. Since

$$\left|\frac{x}{y} - \theta_0\right| > 2,$$

the assumption $|Q_0(x, y)| < |Q_0(1, 0)| = |a|$ and (4.3.3) imply

$$2y^2\left|\frac{x}{y} + \frac{1}{\eta_0}\right| < y^2\left|\frac{x}{y} - \theta_0\right|\left|\frac{x}{y} + \frac{1}{\eta_0}\right| < 1.$$

Hence

$$\left|\frac{x}{y} + \frac{1}{\eta_0}\right| < \frac{1}{2y^2}.$$

As in the argument for θ_0, write the reduced form of $-x/y$ as p'/q'. Then $q' \leq y$, and p'/q' is a convergent of $1/\eta_0$. Since $p'/q' > 0$, there exists $m \geq 1$ such that

$$\frac{p'}{q'} = [0; a_{-1}, a_{-2}, \ldots, a_{-m}].$$

Writing the right-hand side as p'_m/q'_m, we have

$$\begin{bmatrix} p'_m & * \\ q'_m & * \end{bmatrix} = F_0 F_{a_{-1}} F_{a_{-2}} \cdots F_{a_{-m}}.$$

Using the first column, then taking the transpose and the inverse of this matrix, we obtain

$$(-1)^{m+1}\begin{bmatrix} q'_m & * \\ -p'_m & * \end{bmatrix} = F_0^{-1} F_{a_{-1}}^{-1} F_{a_{-2}}^{-1} \cdots F_{a_{-m}}^{-1}.$$

Moving F_0^{-1} to the other side and applying both sides to $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ gives

$$(-1)^{m+1}\begin{bmatrix} -p'_m \\ q'_m \end{bmatrix} = F_{a_{-1}}^{-1} F_{a_{-2}}^{-1} \cdots F_{a_{-m}}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Applying Q_0 to both sides yields

$$Q_0((-1)^m p'_m, (-1)^{m+1} q'_m) = Q_{-m}(1, 0).$$

Since every quadratic form satisfies $Q(u, -v) = Q(-u, v)$, for every $m \in \mathbb{Z}_{\geq 1}$ we have

$$Q_0(p'_m, -q'_m) = Q_0(-p'_m, q'_m) = Q_0((-1)^m p'_m, (-1)^{m+1} q'_m) = Q_{-m}(1, 0).$$

Now $x/y = -p'_m/q'_m$, and p'_m/q'_m is in lowest terms. Hence there exists $d \in \mathbb{Z}_{\geq 1}$ such that either $(x, y) = (-dp'_m, dq'_m)$ or $(x, y) = (dp'_m, -dq'_m)$. In either case,

$$|Q_0(x, y)| = d^2|Q_0(-p'_m, q'_m)| \geq |Q_0(-p'_m, q'_m)| = |Q_{-m}(1, 0)|.$$

The proof is complete. □

We are now ready to prove Theorem 4.3.3. With the preparations in place, the proof is short.

Proof of Theorem 4.3.3. For $Q = Q_0$, we must show

$$\sup_{h \in \mathbb{Z}}([a_h; a_{h+1}, \ldots] + [0; a_{h-1}, a_{h-2}, \ldots]) = \frac{\sqrt{D(Q)}}{\inf_{(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |Q(x, y)|}.$$

By Proposition 4.3.9, it suffices to prove the equality of extended real numbers

$$\frac{\sqrt{D(Q_0)}}{\inf_{h \in \mathbb{Z}} |Q_h(1, 0)|} = \frac{\sqrt{D(Q_0)}}{\inf_{(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |Q_0(x, y)|}.$$

If both denominators are 0, both sides are read as ∞. In all cases, the equality follows from Proposition 4.3.10. □

We close this section by commenting on the inclusion relation between $\mathcal{L}$ and $\mathcal{M}$. Corollary 4.3.4 shows that $\mathcal{L} \subset \mathcal{M}$. This inclusion is known to be proper, as was proved in various works including [**Fre68**]. Concrete values that belong to $\mathcal{M}$ but not to $\mathcal{L}$ are still actively studied; see, for example, [**LMMR20**] for a detailed account of known constructions in $\mathcal{M} \setminus \mathcal{L}$.

4. Markov Constants of Quadratic Forms with Rational Coefficients

We finish this chapter by considering $\mathcal{M}(Q)$ in the case where the coefficients a, b, c of $Q(x, y) = ax^2 + bxy + cy^2$ are rational. From the previous section, we know how to compute the Markov constant when Q is canonical reduced. Thus we would like to replace Q by a canonical reduced form Q' satisfying $\mathcal{M}(Q) = \mathcal{M}(Q')$. The existence of such a form is guaranteed by Theorem 4.2.5, and the proof also gives an algorithm to find it. When a, b, c are rational, however, the theory of quadratic irrationals tells us more directly which form Q' should be used.

Let

$$\mathcal{Q}_{\mathbb{Q}} := \{Q(x, y) = ax^2 + bxy + cy^2 \in \mathcal{Q} \mid a, b, c \in \mathbb{Q}\}$$

be the set of quadratic forms in $\mathcal{Q}$ with rational coefficients.

PROPOSITION 4.4.1. *Let $Q(x, y) = ax^2 + bxy + cy^2 \in \mathcal{Q}_{\mathbb{Q}}$. Then $Q(x, 1)$ has two quadratic irrational roots, which are conjugate to each other. Choose one of them and call it α. Suppose that the infinite continued-fraction expansion of α is*

$$\alpha = [a_0; a_1, \dots, a_n, \overline{b_0, \dots, b_{k-1}}]$$

for some $n \geq 0$ and $k \geq 1$. Put

$$\beta := [\overline{b_0, b_1, \dots, b_{k-1}}],$$

and let β' be the quadratic conjugate of β. Define

$$Q'(x, y) := (x - \beta y)(x - \beta' y).$$

Then

$$\mathcal{M}(Q) = \mathcal{M}(Q'),$$

and Q' is canonical reduced.

PROOF. By Theorem 2.5.5, the numbers α and β are equivalent. Thus there exists $A \in GL(2, \mathbb{Z})$ such that $\beta = A\alpha$. Consider the quadratic form QA^{-1}. By Lemma 4.2.6, the polynomial $QA^{-1}(x, 1)$ has β as a root. Since $QA^{-1}(x, y)$ has rational coefficients, the roots of $QA^{-1}(x, 1)$ are β and β'. Therefore there exists $a' \in \mathbb{Q} \setminus \{0\}$ such that

$$QA^{-1}(x, y) = a'(x - \beta y)(x - \beta' y).$$

Since β is reduced, we have $\beta > 1$ and $-1 < \beta' < 0$. Hence QA^{-1} is canonical reduced. By Corollary 4.3.1,

$$\mathcal{M}(Q) = \mathcal{M}(QA^{-1}).$$

Moreover $Q' = (1/a')QA^{-1}$. Canonical reducedness is preserved under multiplication by a nonzero scalar, and the Markov constant is unchanged by such a scalar multiple. Hence Q' is canonical reduced and $\mathcal{M}(Q') = \mathcal{M}(Q)$. □

Proposition 4.4.1 determines $\mathcal{M}(Q)$ from one of the two roots of $Q(x, 1)$ or $Q(1, y)$. One might therefore wonder whether the Markov constant should depend on the information of both irrational roots. In the present case, however, the form has rational coefficients: if α is an irrational root of $Q(x, 1)$, then its quadratic conjugate α' is the other root. Thus, in this setting, the Markov constant is essentially determined by a single irrational number. In this case there is a clean correspondence between the Lagrange constant and the Markov constant.

THEOREM 4.4.2. *Let $Q \in \mathcal{Q}_{\mathbb{Q}}$, and let α be one of the quadratic irrational roots of $Q(x, 1) = 0$. Then*

$$\mathcal{M}(Q) = \mathcal{L}(\alpha).$$

PROOF. Suppose that the infinite continued-fraction expansion of α is

$$\alpha = [a_0; a_1, \ldots, a_n, \overline{b_0, \ldots, b_{k-1}}]$$

for some $n \geq 0$ and $k \geq 1$. Put $\beta = [\overline{b_0, b_1, \ldots, b_{k-1}}]$ and $Q'(x, y) = (x - \beta y)(x - \beta' y)$. By Proposition 4.4.1 and Theorem 4.3.3, if

$$\mathbf{b} = (\ldots, b_0, b_1, \ldots, b_{k-1}, b_0, b_1, \ldots, b_{k-1}, b_0, b_1, \ldots, b_{k-1}, \ldots)$$

is the bi-infinite sequence obtained by repeating the period in both directions, then

$$\mathcal{M}(Q) = \mathcal{M}(Q') = \mathcal{S}(\mathbf{b}).$$

On the other hand, Proposition 3.3.1 and Theorem 3.3.3 give

$$\mathcal{L}(\alpha) = \mathcal{L}(\beta) = \mathcal{S}(\mathbf{b}).$$

Therefore $\mathcal{M}(Q) = \mathcal{L}(\alpha)$. □

Let us compute the Markov constant of a quadratic form analogous to Example 3.3.7.

EXAMPLE 4.4.3. We compute the Markov constant $\mathcal{M}(Q)$ of the rational-coefficient quadratic form $Q(x, y) = x^2 - 2xy - 2y^2$. The roots of $Q(x, 1) = x^2 - 2x - 2$ are $x = 1 \pm \sqrt{3}$. Thus we may take $\alpha = 1 + \sqrt{3}$ as one quadratic irrational root of $Q(x, 1)$. Example 3.3.7 gives $\alpha = [\overline{2, 1}]$; hence the period is $(2, 1)$ and its length is $k = 2$. The purely periodic continued fraction determined by this period is $\beta = [\overline{2, 1}]$. In this case $\beta = \alpha$, and its quadratic conjugate is $\beta' = 1 - \sqrt{3}$. Since $\beta > 1$ and $-1 < \beta' < 0$, the form Q is canonical reduced. By Proposition 4.4.1 and Theorem 4.3.3, for the bi-infinite sequence $\mathbf{b} = (\ldots, 2, 1, 2, 1, 2, 1, 2, 1, \ldots)$ obtained by repeating the period $(2, 1)$ in both directions, we have $\mathcal{M}(Q) = \mathcal{S}(\mathbf{b})$. On the other hand, the previous example showed that $\mathcal{L}(1 + \sqrt{3}) = 2\sqrt{3}$. Therefore Theorem 4.4.2 gives $\mathcal{M}(Q) = \mathcal{L}(\alpha) = 2\sqrt{3}$.

In fact, Example 3.1.5 and Example 4.1.5 also correspond to each other through Theorem 4.4.2. The reader may check this directly.

We conclude the chapter by summarizing the relation between the Markov spectrum and the Lagrange spectrum obtained so far. Corollary 4.3.4 shows that in general $\mathcal{L} \subset \mathcal{M}$, and it is known that the reverse inclusion does not hold. However, appropriate restrictions of the two sets are exactly the same.

COROLLARY 4.4.4. *Let*

$$\mathcal{L}_2 := \{\mathcal{L}(\alpha) \mid \alpha \in I_2\}, \qquad \mathcal{M}_{\mathbb{Q}} := \{\mathcal{M}(Q) \mid Q \in \mathcal{Q}_{\mathbb{Q}}\},$$

and

$$\mathcal{S}_{\text{period}} := \{\mathcal{S}(\mathbf{b}) \mid \mathbf{b} \textit{ is a periodic bi-infinite sequence}\}.$$

Then

$$\mathcal{L}_2 = \mathcal{M}_{\mathbb{Q}} = \mathcal{S}_{\text{period}}.$$

PROOF. This follows from Corollary 3.3.4 and Theorem 4.4.2. □

From the next chapter onward, we study values in $\mathcal{L}_2$ and $\mathcal{M}_{\mathbb{Q}}$ that admit special descriptions in terms of generalized Markov numbers.

Part 2

Generalized Markov Numbers

CHAPTER 5

Generalized Markov Equations and Generalized Markov Numbers

In this chapter we introduce generalized Markov equations and generalized Markov numbers, which form the starting point of the second part of the text. Up to Chapter 4, we studied the Lagrange spectrum and the Markov spectrum. The purpose of the present chapter is to prepare the arithmetic objects that will later be used to describe a discrete family of values in that theory. We first define the generalized Markov equation and record its basic properties. We then construct generalized Markov trees and organize all positive integer solutions. After that, we introduce fraction labels through the correspondence with the Farey tree. Finally, we define characteristic numbers, which will later be used to describe generalized Cohn matrices and generalized discrete Markov spectra.

The material in this chapter is based on papers by the author and collaborators [**GM23a, GM23b, GMS24, BG25**]. The papers [**GM23b, GMS24**] give proofs only in the symmetric case $k_1 = k_2 = k_3$; here we rewrite the arguments in full generality. The paper [**BG25**] treats a more general framework coming from cluster algebra theory; here we specialize those arguments so that the discussion remains within elementary number theory.

1. Definitions and Basic Properties

DEFINITION 5.1.1. For $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$, the (k_1, k_2, k_3)-*generalized Markov equation* is

$$x^2 + y^2 + z^2 + k_1yz + k_2zx + k_3xy = (3 + k_1 + k_2 + k_3)xyz. \tag{5.1.1}$$

A permutation of a positive integer solution (x, y, z) of this equation is called a (k_1, k_2, k_3)-*generalized Markov triple*. A positive integer that occurs in a (k_1, k_2, k_3)-generalized Markov triple is called a (k_1, k_2, k_3)-*generalized Markov number*. When $k_1 = k_2 = k_3 = 0$, we simply call them Markov triples and Markov numbers.

Since these names are long, we henceforth abbreviate "generalized Markov" to "GM". One point should be kept in mind. A (k_1, k_2, k_3)-GM triple includes not only a positive integer solution of the (k_1, k_2, k_3)-GM equation, but also its permutations. If $k_1 = k_2 = k_3$, then the equation is symmetric in the three variables x, y, z, and hence every permutation of a positive integer solution is again a positive integer solution. In that case there is no difference between the two notions. If the parameters are not all equal, however, a permutation of a positive integer solution need not be a positive integer solution.

Let us first discuss basic properties of the positive integer solutions of the GM equation. At this stage the order of the three components is taken into account, so we are not yet speaking about GM triples. We begin with an algorithm that enumerates all positive integer solutions. Define a tree $\mathbb{T}(k_1, k_2, k_3)$ whose vertices are triples of positive integers as follows.

(1) The initial vertex is $(1, 1, 1)$.
(2) The triple $(1, 1, 1)$ has the following three children: $(k_1 + 2, 1, 1)$, $(1, k_2 + 2, 1)$, and $(1, 1, k_3 + 2)$.
(3) At every vertex other than the initial vertex, the generation rule is as follows.
   (i) If a is the largest component of (a, b, c), then (a, b, c) has the following two children:

$$\left(a, \frac{a^2 + k_2ac + c^2}{b}, c\right), \quad \left(a, b, \frac{a^2 + k_3ab + b^2}{c}\right).$$

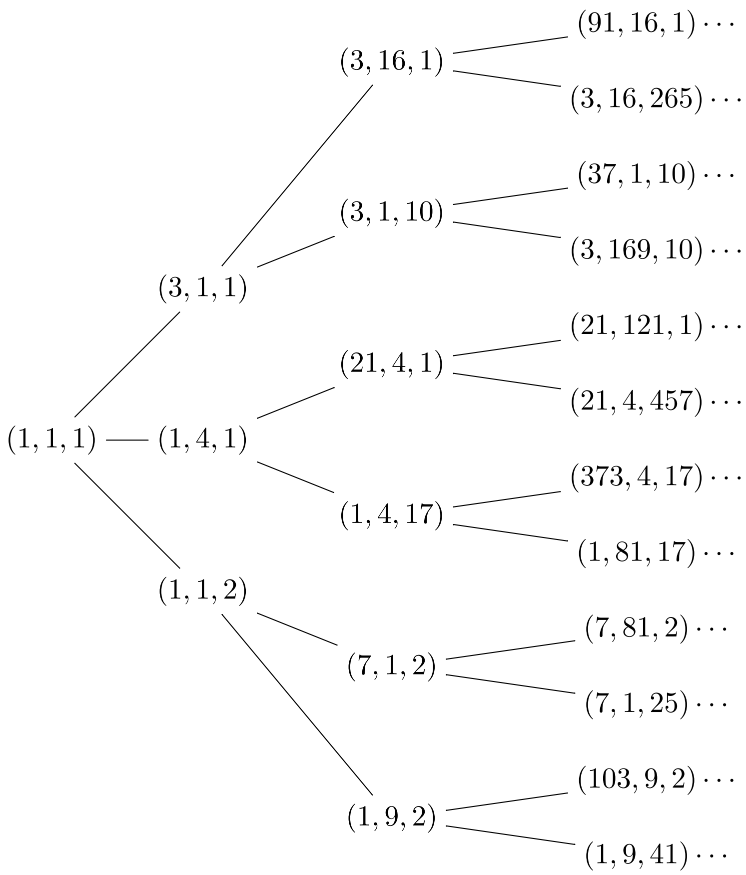
(ii) If b is the largest component of (a,b,c), then (a,b,c) has the following two children:

$$\left(\frac{b^2+k_1bc+c^2}{a}, b, c\right), \quad \left(a, b, \frac{a^2+k_3ab+b^2}{c}\right).$$

(iii) If c is the largest component of (a,b,c), then (a,b,c) has the following two children:

$$\left(\frac{b^2+k_1bc+c^2}{a}, b, c\right), \quad \left(a, \frac{a^2+k_2ac+c^2}{b}, c\right).$$

EXAMPLE 5.1.2. For $k_1 = 1, k_2 = 2, k_3 = 0$, the first few vertices of $\mathbb{T}(1,2,0)$ are as follows:



We have the following theorem.

THEOREM 5.1.3. *Every positive integer solution of the (k_1,k_2,k_3)-GM equation appears exactly once in $\mathbb{T}(k_1,k_2,k_3)$.*

We prepare the proof with the following proposition.

PROPOSITION 5.1.4. *Suppose that $(x,y,z) = (a,b,c)$ is a positive integer solution of (5.1.1). Then*

$$\left(\frac{b^2+k_1bc+c^2}{a}, b, c\right), \quad \left(a, \frac{a^2+k_2ac+c^2}{b}, c\right), \quad \left(a, b, \frac{a^2+k_3ab+b^2}{c}\right)$$

are also positive integer solutions of (5.1.1).

PROOF. It suffices to prove the assertion for $\left(\frac{b^2+k_1bc+c^2}{a}, b, c\right)$. Positivity is clear, so it remains to prove that this is a triple of integers and satisfies (5.1.1). Since (a,b,c) is a solution of (5.1.1), we have

$$\frac{b^2+k_1bc+c^2}{a} = (3+k_1+k_2+k_3)bc - a - k_3b - k_2c.$$

Thus $\left(\dfrac{b^2+k_1bc+c^2}{a}, b, c\right)$ is a triple of integers. To make the following computation easier to read, put $A := (3+k_1+k_2+k_3)bc - a - k_3b - k_2c$. We show that (A, b, c) is a solution of (5.1.1).

The sum and product of a and A are

$$\begin{aligned} a + A &= (3 + k_1 + k_2 + k_3)bc - k_3b - k_2c, \\ a \cdot A &= b^2 + k_1bc + c^2. \end{aligned}$$

By the relation between roots and coefficients, a and A are the two roots of

$$X^2 - \{(3 + k_1 + k_2 + k_3)bc - k_3b - k_2c\}X + b^2 + k_1bc + c^2 = 0.$$

Substituting $X = A$ into this quadratic equation and rearranging, we obtain

$$A^2 + b^2 + c^2 + k_3Ab + k_1bc + k_2cA = (3 + k_1 + k_2 + k_3)Abc.$$

This is exactly (5.1.1) with $(x, y, z) = (A, b, c)$. □

We call the three operations

$$\begin{aligned} (a, b, c) &\mapsto \left(\frac{b^2 + k_1bc + c^2}{a}, b, c\right), \\ (a, b, c) &\mapsto \left(a, \frac{a^2 + k_2ac + c^2}{b}, c\right), \\ (a, b, c) &\mapsto \left(a, b, \frac{a^2 + k_3ab + b^2}{c}\right) \end{aligned}$$

the *first, second, and third Vieta jumps*, respectively. Each Vieta jump is an involution, namely applying the same operation again returns the original triple. We next determine the solutions that contain two equal components.

LEMMA 5.1.5. *The positive integer solutions of* (5.1.1) *that contain equal components are only*

$$(1, 1, 1),\ (k_1 + 2, 1, 1),\ (1, k_2 + 2, 1),\ (1, 1, k_3 + 2).$$

PROOF. Let (a, b, c) be a positive integer solution of (5.1.1) that contains equal components. We prove only the case $a = b$. Substituting $(x, y, z) = (a, a, c)$ into (5.1.1), we obtain

$$(2 + k_3)a^2 + c^2 + (k_1 + k_2)ac = (3 + k_1 + k_2 + k_3)a^2c.$$

Therefore

$$c = \frac{1}{2}\Big(a^2k_3 + a^2k_1 + a^2k_2 + 3a^2 - ak_1 - ak_2 \pm a\sqrt{(ak_3 + (a-1)k_1 + (a-1)k_2 + 3a)^2 - 4(k_3 + 2)}\Big).$$

Put $k := ak_3 + (a-1)k_1 + (a-1)k_2 + 3a > 0$. Since c is an integer, the expression under the square root must be a square. Hence there exists a positive integer l such that $l^2 = k^2 - 4(k_3+2)$. Because $a \geq 1$, we have $k \geq k_3 + 3$, and hence $k + l > k_3 + 2$. From $(k+l)(k-l) = 4(k_3+2)$, it follows that $1 \leq k - l \leq 3$, and so

$$(k - l, k + l) = (1, 4(k_3 + 2)),\ (2, 2(k_3 + 2)),\ \left(3, \frac{4(k_3 + 2)}{3}\right).$$

Since $k = ((k + l) + (k - l))/2$ must be an integer, the first and the third possibilities are impossible. In the case $(k-l, k+l) = (2, 2(k_3+2))$, we obtain $k = k_3 + 3$ and $l = k_3 + 1$, and hence obtain $(a, a, c) = (1, 1, 1)$ or $(1, 1, k_3 + 2)$. The cases $a = c$ and $b = c$ are proved in the same way, with the corresponding parameter k_2 or k_1 in place of k_3. □

We call the four triples below

$$(1, 1, 1),\ (k_1 + 2, 1, 1),\ (1, k_2 + 2, 1),\ (1, 1, k_3 + 2)$$

singular. We call every other positive integer solution of (5.1.1) *nonsingular*.

Proposition 5.1.6. *Let $(x,y,z)=(a,b,c)$ be a nonsingular positive integer solution of (5.1.1), and assume that $a>b>c$. Then the following hold:*

(1) $\dfrac{a^2+k_2ac+c^2}{b}>a(>c)$, *(2)* $\dfrac{a^2+k_3ab+b^2}{c}>a(>b)$, *(3)* $b>\dfrac{b^2+k_1bc+c^2}{a}$.

Proof. We prove (1). Since

$$\frac{a^2+k_2ac+c^2}{b}-a=\frac{a^2+k_2ac+c^2-ab}{b}>\frac{a^2+k_2ac+c^2-a^2}{b}=\frac{c^2+k_2ac}{b}>0,$$

the claim follows. The proof of (2) is the same. We now prove (3). Put

$$\begin{aligned} f(x) &:= (x-a)\left(x-\frac{b^2+k_1bc+c^2}{a}\right) \\ &= x^2-((3+k_1+k_2+k_3)bc-k_3b-k_2c)x+(b^2+c^2+k_1bc). \end{aligned}$$

It suffices to prove $f(b)<0$, that is, $f(b)=(2+k_3)b^2-(3+k_1+k_2+k_3)b^2c+(k_1+k_2)bc+c^2<0$. Consider the function on $\mathbb{R}^2$

$$g(y,z)=(2+k_3)y^2-(3+k_1+k_2+k_3)y^2z+(k_1+k_2)yz+z^2. \tag{5.1.2}$$

Then $g(b,c)=f(b)$. Its partial derivative in the y-direction is

$$\frac{\partial g}{\partial y}=2(2+k_3)y-2(3+k_1+k_2+k_3)yz+(k_1+k_2)z.$$

Under the condition $y>z\geq 1$, we have

$$\begin{aligned} \frac{\partial g}{\partial y}(y,z) &< 2(2+k_3)y-2(3+k_1+k_2+k_3)yz+(k_1+k_2)y \\ &= -y((6z-4)+k_3(2z-2)+k_1(2z-1)+k_2(2z-1)) \\ &< -((6z-4)+k_3(2z-2)+k_1(2z-1)+k_2(2z-1))<0. \end{aligned}$$

Similarly, the partial derivative in the z-direction is

$$\frac{\partial g}{\partial z}=-(3+k_1+k_2+k_3)y^2+(k_1+k_2)y+2z,$$

and under $y>z\geq 1$ we have

$$\begin{aligned} \frac{\partial g}{\partial z}(y,z) &< -(3+k_1+k_2+k_3)y^2+(k_1+k_2)y+2y \\ &< -(3+k_1+k_2+k_3)y^2+(k_1+k_2)y^2+2y^2 \\ &= -y^2(1+k_3)<0. \end{aligned}$$

Thus, on the region $y>z\geq 1$, the function $g(y,z)$ is strictly decreasing in both the y- and z-directions. Now move from $(1,1)$ to (b,c) along the path that first increases y from 1 to b while keeping $z=1$, and then increases z from 1 to c while keeping $y=b$. Applying the above monotonicity along this path and using $g(1,1)=0$, we obtain $g(b,c)=f(b)<0$. □

In Proposition 5.1.6 we assumed $a>b>c$, but this assumption is not essential.

Corollary 5.1.7. *Let $(x,y,z)=(a,b,c)$ be a nonsingular positive integer solution of (5.1.1). Let (a',b,c), respectively (a,b',c) and (a,b,c'), be the first, respectively second and third, Vieta jump of (a,b,c).*

(1) *If a is the largest component of (a,b,c), then a' is not the largest component of (a',b,c), while b' is the largest component of (a,b',c) and c' is the largest component of (a,b,c').*
(2) *If b is the largest component of (a,b,c), then a' is the largest component of (a',b,c), while b' is not the largest component of (a,b',c) and c' is the largest component of (a,b,c').*
(3) *If c is the largest component of (a,b,c), then a' is the largest component of (a',b,c) and b' is the largest component of (a,b',c), while c' is not the largest component of (a,b,c').*

PROOF. The case $a > b > c$ is Proposition 5.1.6. The other possible orders are proved by the same Vieta-jump comparison, keeping the coefficients k_1, k_2, k_3 in their original positions. Since the generalized equation is not symmetric under arbitrary permutations of the variables when (k_1, k_2, k_3) is not constant, one cannot simply permute the proof of Proposition 5.1.6. Instead, for each order one rewrites the equation as a quadratic equation in the component being changed and repeats the same monotonicity argument for the corresponding two remaining variables. This shows that the jump at the largest component is the only jump that does not produce a new largest component, whereas a jump at either of the other two components does produce a new largest component. The three assertions follow. □

REMARK 5.1.8. From Corollary 5.1.7 and the generation rule for $\mathbb{T}(k_1, k_2, k_3)$, we obtain the following facts for a nonsingular triple (a, b, c) in $\mathbb{T}(k_1, k_2, k_3)$.

(i) If a is the largest component of (a, b, c), then the parent of (a, b, c) is

$$\left(\frac{b^2 + k_1 bc + c^2}{a}, b, c\right).$$

(ii) If b is the largest component of (a, b, c), then the parent of (a, b, c) is

$$\left(a, \frac{a^2 + k_2 ac + c^2}{b}, c\right).$$

(iii) If c is the largest component of (a, b, c), then the parent of (a, b, c) is

$$\left(a, b, \frac{a^2 + k_3 ab + b^2}{c}\right).$$

Moreover, every nonsingular triple in $\mathbb{T}(k_1, k_2, k_3)$ has a largest component that is smaller than the largest components of its children. Hence a singular triple cannot be a child of a nonsingular triple. It follows that each singular triple appears exactly once in $\mathbb{T}(k_1, k_2, k_3)$. The statements (i), (ii), and (iii) also hold for the singular triples other than $(1, 1, 1)$.

Consequently, for every vertex (a, b, c) of $\mathbb{T}(k_1, k_2, k_3)$, the three vertices adjacent to it are obtained by replacing exactly one of the components a, b, c by another number.

We now prove Theorem 5.1.3.

PROOF OF THEOREM 5.1.3. By Proposition 5.1.4 and the fact that $(1, 1, 1)$ is a positive integer solution of (5.1.1), every vertex of $\mathbb{T}(k_1, k_2, k_3)$ is a positive integer solution of (5.1.1). We prove that there are no positive integer solutions other than these vertices. Let $(x, y, z) = (a, b, c)$ be a nonsingular positive integer solution of (5.1.1). By Corollary 5.1.7, among the Vieta jumps of (a, b, c) there is a unique one that decreases the largest component. This operation can be repeated as long as the solution remains nonsingular. All solutions that occur during this process are positive integer solutions, so after finitely many steps we reach a singular solution. By Lemma 5.1.5, when a nonsingular solution moves to a singular one, that singular solution is one of $(k_1 + 2, 1, 1)$, $(1, k_2 + 2, 1)$, and $(1, 1, k_3 + 2)$. By Remark 5.1.8, every Vieta jump of a triple in $\mathbb{T}(k_1, k_2, k_3)$ again belongs to $\mathbb{T}(k_1, k_2, k_3)$. Therefore, by reversing the above process, (a, b, c) appears as a vertex of $\mathbb{T}(k_1, k_2, k_3)$.

It remains to prove uniqueness. Which of the first, second, and third Vieta jumps moves a triple toward the initial vertex is determined solely by the order of the components of (a, b, c). Thus the path from (a, b, c) back to $(1, 1, 1)$ is independent of the position at which (a, b, c) might appear in the tree. If (a, b, c) appeared in two different positions, then these two occurrences would give two distinct paths to the root, contradicting the fact that the parent at each non-root vertex is uniquely determined by Remark 5.1.8. □

We record an important corollary of Theorem 5.1.3.

COROLLARY 5.1.9. *For every positive integer solution* (a, b, c) *of* (5.1.1)*, any two of* a, b, c *are relatively prime.*

Proof. The assertion is clear for $(a, b, c) = (1, 1, 1)$. We prove only that a and b are relatively prime. Rewrite (5.1.1) as

$$z^2 = (3 + k_1 + k_2 + k_3)xyz - x^2 - y^2 - k_1yz - k_2zx - k_3xy,$$

and substitute $(x, y, z) = (a, b, c)$. Suppose that a and b have a common divisor $d \neq 1$, and let d' be a prime divisor of d. Reducing the displayed equation modulo d' gives $c^2 \equiv 0 \pmod{d'}$, because d' divides both a and b. Hence d' divides c, and so d' is a common divisor of a, b, c.

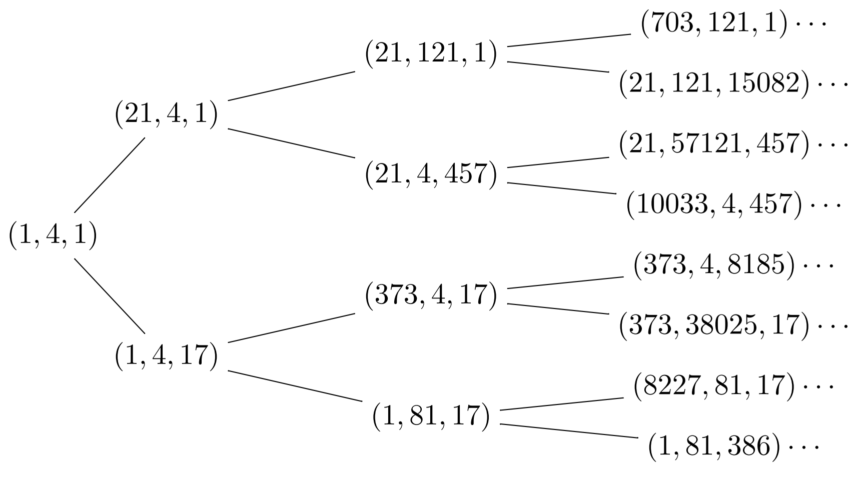
By Proposition 5.1.4, the adjacent triple (a', b', c') in $\mathbb{T}(k_1, k_2, k_3)$ whose largest component is smaller than $\max\{a, b, c\}$ is obtained by replacing the largest component by the other root of the corresponding quadratic equation. If the first component is replaced, then the two roots have sum $(3 + k_1 + k_2 + k_3)bc - k_2c - k_3b$, which is divisible by d'; since the old first component is also divisible by d', the new one is divisible by d' as well. The arguments for the second and third components are identical. Hence common divisibility by d' is preserved along the descent. Repeating the operation, we conclude that d' is eventually a common divisor of $(1, 1, 1)$, which is impossible. Therefore $d = 1$, and a and b are relatively prime. □

2. Generalized Markov Trees

The tree $\mathbb{T}(k_1, k_2, k_3)$ is economical from the point of view of enumerating all positive integer solutions of the (k_1, k_2, k_3)-GM equation. However, when we later compare it with the matrix theory, it will often be more convenient to decompose this tree into several binary trees. We therefore introduce new binary trees.

Definition 5.2.1. The full subtrees of $\mathbb{T}(k_1, k_2, k_3)$ whose initial vertices are respectively $(k_1 + 2, 1, 1)$, $(1, k_2 + 2, 1)$, and $(1, 1, k_3 + 2)$ are called the *first, second, and third branches* of $\mathbb{T}(k_1, k_2, k_3)$ and are denoted by $\mathbb{T}_1(k_1, k_2, k_3)$, $\mathbb{T}_2(k_1, k_2, k_3)$, and $\mathbb{T}_3(k_1, k_2, k_3)$.

Example 5.2.2. The first few vertices of $\mathbb{T}_2(1, 2, 0)$ are as follows:



By definition, each $\mathbb{T}_i(k_1, k_2, k_3)$ is a complete binary tree. In the theory below, however, we will actually use the following complete binary trees, obtained by rotating the components at each vertex so that the newly produced component is written in the middle. The branches above will be used to prove that the vertices of the binary trees defined below enumerate all positive integer solutions.

Definition 5.2.3. Let $\mathfrak{S}_3$ be the symmetric group of degree 3, acting on the left on $\{1, 2, 3\}$. For $\sigma \in \mathfrak{S}_3$, define the (k_1, k_2, k_3, σ)-*generalized Markov tree* (or simply the *GM tree*) $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$ as follows.

(1) The initial vertex is

$$((1, \sigma(1)), (k_{\sigma(2)} + 2, \sigma(2)), (1, \sigma(3))).$$

(2) For every vertex $((a,\alpha),(b,\beta),(c,\gamma))$, define the following two children, distinguishing the left child from the right child:

$$((a,\alpha),(b,\beta),(c,\gamma))$$

$$\left((a,\alpha),\left(\tfrac{a^2+k_\gamma ab+b^2}{c},\gamma\right),(b,\beta)\right) \qquad \left((b,\beta),\left(\tfrac{b^2+k_\alpha bc+c^2}{a},\alpha\right),(c,\gamma)\right)$$

EXAMPLE 5.2.4. The first few vertices of $\mathrm{M}\mathbb{T}(1,2,0,\mathrm{id})$ are as follows:

- $((1,1),(4,2),(1,3))$
  - $((4,2),(21,1),(1,3))$
    - $((21,1),(121,2),(1,3))$
      - $((121,2),(703,1),(1,3))\cdots$
      - $((21,1),(15082,3),(121,2))\cdots$
    - $((4,2),(457,3),(21,1))$
      - $((457,3),(57121,2),(21,1))\cdots$
      - $((4,2),(10033,1),(457,3))\cdots$
  - $((1,1),(17,3),(4,2))$
    - $((17,3),(373,1),(4,2))$
      - $((373,1),(8185,3),(4,2))\cdots$
      - $((17,3),(38025,2),(373,1))\cdots$
    - $((1,1),(81,2),(17,3))$
      - $((81,2),(8227,1),(17,3))\cdots$
      - $((1,1),(386,3),(81,2))\cdots$

Each vertex of this tree consists of three pairs, hence of six entries in total. The first entry of each pair is a GM number, while the second entry records the position in which that GM number occurs in the original positive integer solution. Since the (k_1,k_2,k_3)-GM equation is not symmetric in x,y,z, it is important to remember which integer corresponds to which of x,y,z. In this tree, the components rotate at each generation. Therefore this positional data must be stored separately from the GM number itself. We henceforth call a pair consisting of a (k_1,k_2,k_3)-GM number and its positional data a *(k_1,k_2,k_3)-GM pair*.

For $((a,h),(b,i),(c,j)) \in \mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$, we call the m-th entry of the n-th pair the (n,m)-entry.

We now show that $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$ is the same tree as a branch of $\mathbb{T}(k_1,k_2,k_3)$, up to the difference caused by rotations. To do this, we first introduce isomorphisms of complete binary trees.

DEFINITION 5.2.5. Let A and B be regarded as complete binary trees, that is, assume that their elements correspond to the vertices of complete binary trees. A bijection $f : A \to B$ is called a *complete binary tree isomorphism* if, for vertices u and v of A, the vertex v is a child of u if and only if $f(v)$ is a child of $f(u)$. If, in addition, A and B are ordered complete binary trees, meaning that the two children of each vertex are distinguished as the left and right children, and if f also preserves the distinction between left and right children, then f is called an *ordered complete binary tree isomorphism.*

Note that $\mathbb{T}_i(k_1,k_2,k_3)$ is an unordered complete binary tree, whereas $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$ is an ordered complete binary tree. We next construct a bijection that gives an unordered complete binary tree isomorphism between $\mathbb{T}_i(k_1,k_2,k_3)$ and $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$.

For $\tau \in \mathfrak{S}_3$, let ${}^{\tau}(a,b,c)$ denote the triple obtained by permuting (a,b,c) so that a, b, and c become the $\tau(1)$-st, $\tau(2)$-nd, and $\tau(3)$-rd components, respectively. For a tree $\mathbb{T}$, let $V(\mathbb{T})$ denote its vertex set. Define a map

$$\pi_\sigma : V(\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)) \to \mathbb{Z}^3_{>0}$$

as follows.

- For the initial vertex, set

$$\pi_\sigma((1,\sigma(1)),(k_{\sigma(2)}+2,\sigma(2)),(1,\sigma(3))) = {}^{\sigma}\!\left(1,k_{\sigma(2)}+2,1\right).$$

- If $\pi_\sigma((a,\alpha),(b,\beta),(c,\gamma)) = {}^\tau(a,b,c)$, then define

$$\pi_\sigma\left((a,\alpha),\left(\frac{a^2+k_\gamma ab+b^2}{c},\gamma\right),(b,\beta)\right) = {}^{\tau\circ(2\ 3)}\left(a,\frac{a^2+k_\gamma ab+b^2}{c},b\right),$$

$$\pi_\sigma\left((b,\beta),\left(\frac{b^2+k_\alpha bc+c^2}{a},\alpha\right),(c,\gamma)\right) = {}^{\tau\circ(1\ 2)}\left(b,\frac{b^2+k_\alpha bc+c^2}{a},c\right).$$

PROPOSITION 5.2.6. *The map π_σ induces the following complete binary tree isomorphisms:*

$$\begin{aligned}
\pi_{(1\ 2)}&\colon \mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 2)) \simeq \mathbb{T}_1(k_1,k_2,k_3),\\
\pi_{(1\ 3\ 2)}&\colon \mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 3\ 2)) \simeq \mathbb{T}_1(k_1,k_2,k_3),\\
\pi_{\mathrm{id}}&\colon \mathrm{M}\mathbb{T}(k_1,k_2,k_3,\mathrm{id}) \simeq \mathbb{T}_2(k_1,k_2,k_3),\\
\pi_{(1\ 3)}&\colon \mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 3)) \simeq \mathbb{T}_2(k_1,k_2,k_3),\\
\pi_{(2\ 3)}&\colon \mathrm{M}\mathbb{T}(k_1,k_2,k_3,(2\ 3)) \simeq \mathbb{T}_3(k_1,k_2,k_3),\\
\pi_{(1\ 2\ 3)}&\colon \mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 2\ 3)) \simeq \mathbb{T}_3(k_1,k_2,k_3).
\end{aligned}$$

PROOF. We prove the assertion for $\sigma = (1\ 2)$; the other cases are identical. We use the following six states. A vertex $((a,\alpha),(b,\beta),(c,\gamma))$ is said to be in the state indexed by $\tau \in \mathfrak{S}_3$ if

$$\pi_{(1\ 2)}((a,\alpha),(b,\beta),(c,\gamma)) = {}^\tau(a,b,c) \quad \text{and} \quad (\alpha,\beta,\gamma) = (\tau(1),\tau(2),\tau(3)).$$

For the six possible values of τ, the corresponding index triple and the largest component of ${}^\tau(a,b,c)$ are as follows:

State	τ	(α,β,γ)	Largest component of ${}^\tau(a,b,c)$
(1)	id	$(1,2,3)$	second component
(2)	$(2\ 3)$	$(1,3,2)$	third component
(3)	$(1\ 2)$	$(2,1,3)$	first component
(4)	$(1\ 3)$	$(3,2,1)$	second component
(5)	$(1\ 2\ 3)$	$(2,3,1)$	third component
(6)	$(1\ 3\ 2)$	$(3,1,2)$	first component

In each row this means precisely that the middle entry b of the displayed vertex is the largest of a,b,c.

Suppose that a vertex is in the state indexed by τ. By the definition of $\pi_{(1\ 2)}$, its left child is in the state indexed by $\tau\circ(2\ 3)$, and its right child is in the state indexed by $\tau\circ(1\ 2)$. Thus the transitions are

State	left child	right child
(1)	(2)	(3)
(2)	(1)	(6)
(3)	(5)	(1)
(4)	(6)	(5)
(5)	(3)	(4)
(6)	(4)	(2)

The initial vertex of $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 2))$ is $((1,2),(k_1+2,1),(1,3))$, which is in state (3) and is mapped by $\pi_{(1\ 2)}$ to $(k_1+2,1,1)$, the initial vertex of $\mathbb{T}_1(k_1,k_2,k_3)$.

Now assume that a noninitial vertex is in one of the six states. Since its middle entry is the largest of a,b,c, Corollary 5.1.7 and Remark 5.1.8 show that replacing either the first or the third entry produces the two children in the corresponding branch of $\mathbb{T}(k_1,k_2,k_3)$. The same conclusion for the initial vertex is checked directly. The formulas defining the two children of $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 2))$ are exactly these two Vieta jumps, written after the rotation that places the newly produced entry in the middle. Hence $\pi_{(1\ 2)}$ sends the two children of each vertex of $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 2))$ to the two children of its image in $\mathbb{T}_1(k_1,k_2,k_3)$.

It follows by induction on the distance from the initial vertex that $\pi_{(1\ 2)}$ gives a complete binary tree isomorphism from $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,(1\ 2))$ onto $\mathbb{T}_1(k_1,k_2,k_3)$. The proof for the other five values of σ is the same. ∎

REMARK 5.2.7. The isomorphism π does not distinguish left and right children. Hence, if $\sigma^* := \sigma \circ (1\ 3)$, then for every σ the two trees $\mathbb{MT}(k_1,k_2,k_3,\sigma)$ and $\mathbb{MT}(k_1,k_2,k_3,\sigma^*)$ are sent by π to the same $\mathbb{T}_i(k_1,k_2,k_3)$.

COROLLARY 5.2.8. *Let*

$$v = ((a,\alpha),(b,\beta),(c,\gamma)) \in \mathbb{MT}(k_1,k_2,k_3,\sigma),$$

and suppose that $\pi_\sigma(v) = {}^\tau(a,b,c)$. *Then*

$$b > \max\{a,c\}, \qquad \alpha = \tau(1), \quad \beta = \tau(2), \quad \gamma = \tau(3).$$

PROOF. Let $v = ((a,\alpha),(b,\beta),(c,\gamma)) \in \mathbb{MT}(k_1,k_2,k_3,\sigma)$ and suppose that $\pi_\sigma(v) = {}^\tau(a,b,c)$. Then one of the six states (1)–(6) appearing in the proof of Proposition 5.2.6 occurs. Checking these six cases gives the assertion. □

COROLLARY 5.2.9. *Let $V_{(k_1,k_2,k_3)}$ be the set of all vertices of the six GM trees $\mathbb{MT}(k_1,k_2,k_3,\sigma)$ with $\sigma \in \mathfrak{S}_3$. For every $v \in V_{(k_1,k_2,k_3)}$, the position at which v appears is unique in the union of these six trees.*

PROOF. First, if $v = ((a,\alpha),(b,\beta),(c,\gamma)) \in V_{(k_1,k_2,k_3)}$, then the same displayed vertex cannot appear more than once in the tree $\mathbb{MT}(k_1,k_2,k_3,\sigma)$ to which v belongs; otherwise the bijectivity of Proposition 5.2.6 would be contradicted. We prove that v cannot appear in two different trees $\mathbb{MT}(k_1,k_2,k_3,\sigma)$ and $\mathbb{MT}(k_1,k_2,k_3,\sigma')$. Whether v is the initial vertex, and if it is not the initial vertex whether it is the left or right child of its parent, can be determined from the data of v itself. Indeed, if $a = c$, then the image of v under the corresponding map π_σ is a singular solution. Since singular solutions occur only at the initial vertices of the branches by Remark 5.1.8, the vertex v itself is the initial vertex. Here Corollary 5.2.8 shows that neither $a = b$ nor $b = c$ can occur. If $a \neq c$, then v is not the initial vertex and has a parent. In this case, according as v is the left child or the right child of its parent, the parent is one of

$$\left((a,\alpha),(c,\gamma),\left(\frac{a^2+k_\beta ac+c^2}{b},\beta\right)\right), \qquad \left(\left(\frac{a^2+k_\beta ac+c^2}{b},\beta\right),(a,\alpha),(c,\gamma)\right).$$

By Corollary 5.2.8, the $(2,1)$-entry is strictly larger than the $(1,1)$- and $(3,1)$-entries. Therefore, if $a < c$ then the parent is the former, and if $a > c$ then the parent is the latter. Thus the parent is uniquely determined. It follows that the path from v to the initial vertex is uniquely determined by v, and the displayed form of the initial vertex is also uniquely determined. Hence v cannot belong to two different trees $\mathbb{MT}(k_1,k_2,k_3,\sigma)$ and $\mathbb{MT}(k_1,k_2,k_3,\sigma')$. □

The next theorem says that $\mathbb{MT}(k_1,k_2,k_3,\sigma)$ enumerates GM triples whose second component is the largest one.

THEOREM 5.2.10. *Let (a,b,c) be a (k_1,k_2,k_3)-GM triple satisfying $b > \max\{a,c\}$, and suppose that for $\tau \in \mathfrak{S}_3$ the triple ${}^\tau(a,b,c)$ is a solution of the (k_1,k_2,k_3)-GM equation. Then there exists a unique $\sigma \in \mathfrak{S}_3$ and unique vertices v and v^*, where $\sigma^* := \sigma \circ (1\ 3)$, such that $v \in \mathbb{MT}(k_1,k_2,k_3,\sigma)$, $v^* \in \mathbb{MT}(k_1,k_2,k_3,\sigma^*)$, $v = ((a,\tau(1)),(b,\tau(2)),(c,\tau(3)))$, and $v^* = ((c,\tau(3)),(b,\tau(2)),(a,\tau(1)))$. Moreover, the position of v^* in $\mathbb{MT}(k_1,k_2,k_3,\sigma^*)$ is the mirror image of the position of v in $\mathbb{MT}(k_1,k_2,k_3,\sigma)$, obtained by interchanging left and right at every level.*

PROOF. Since ${}^\tau(a,b,c) \neq (1,1,1)$, Theorem 5.1.3 implies that ${}^\tau(a,b,c)$ belongs to one of $\mathbb{T}_1(k_1,k_2,k_3)$, $\mathbb{T}_2(k_1,k_2,k_3)$, and $\mathbb{T}_3(k_1,k_2,k_3)$. Suppose that ${}^\tau(a,b,c)$ belongs to $\mathbb{T}_1(k_1,k_2,k_3)$; the other cases are handled in the same way. By Proposition 5.2.6, there exist unique vertices v_1 of $\mathbb{MT}(k_1,k_2,k_3,(1\ 2))$ and v_2 of $\mathbb{MT}(k_1,k_2,k_3,(1\ 3\ 2))$ such that $\pi_{(1\ 2)}(v_1) = {}^\tau(a,b,c)$ and $\pi_{(1\ 3\ 2)}(v_2) = {}^\tau(a,b,c)$. The triple ${}^\tau(a,b,c)$ can be displayed in the following six ways:

$${}^\tau(a,b,c), \quad {}^{\tau\circ(1\ 3)}(c,b,a), \quad {}^{\tau\circ(1\ 2)}(b,a,c), \quad {}^{\tau\circ(2\ 3)}(a,c,b), \quad {}^{\tau\circ(1\ 2\ 3)}(b,c,a), \quad {}^{\tau\circ(1\ 3\ 2)}(c,a,b).$$

Since $b > \max\{a,c\}$ by assumption, Corollary 5.2.8 leaves the following two cases:

(1) $v_1 = ((a,\tau(1)),(b,\tau(2)),(c,\tau(3)))$, $\pi_{(1\ 2)}(v_1) = {}^\tau(a,b,c)$, and
$v_2 = ((c,\tau\circ(1\ 3)(1)),(b,\tau\circ(1\ 3)(2)),(a,\tau\circ(1\ 3)(3)))$, $\pi_{(1\ 3\ 2)}(v_2) = {}^{\tau\circ(1\ 3)}(c,b,a)$.

(2) $v_1 = ((c, \tau \circ (1\ 3)(1)), (b, \tau \circ (1\ 3)(2)), (a, \tau \circ (1\ 3)(3)))$, $\pi_{(1\ 2)}(v_1) = {}^{\tau\circ(1\ 3)}(c, b, a)$, and $v_2 = ((a, \tau(1)), (b, \tau(2)), (c, \tau(3)))$, $\pi_{(1\ 3\ 2)}(v_2) = {}^{\tau}(a, b, c)$.

By Corollary 5.2.9, the same displayed vertex cannot occur in two different GM trees. Hence v_1 and v_2 cannot have the same displayed form, and exactly one of (1) and (2) occurs. In case (1), take $\sigma = (1\ 2)$, $\sigma^* = (1\ 3\ 2)$, $v = v_1$, and $v^* = v_2$. In case (2), take $\sigma = (1\ 3\ 2)$, $\sigma^* = (1\ 2)$, $v = v_2$, and $v^* = v_1$. Corollary 5.2.9 also shows that no vertex with the same displayed form as v or v^* appears in any other tree. It remains to show that the position of v^* in $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma^*)$ is obtained from the position of v in $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$ by interchanging left and right at every level. This follows by induction on the distance from the initial vertex. Indeed, the initial vertices of the two trees are obtained from one another by interchanging the first and third pairs. If two vertices are related in this way, then the left child of one is related to the right child of the other, and the right child of one is related to the left child of the other, by the defining formulas for the two children. ☐

3. Farey Trees and Fraction Labels

We next introduce labels of Markov numbers by fractions. For this purpose, we first define Farey triples and the Farey tree.

Definition 5.3.1. For two fractions $\frac{a}{b}$ and $\frac{c}{d}$, write $\det(\frac{a}{b}, \frac{c}{d})$ for $ad - bc$. A triple $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ is called a *Farey triple* if it satisfies the following conditions:

(1) Each of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ is a reduced fraction.

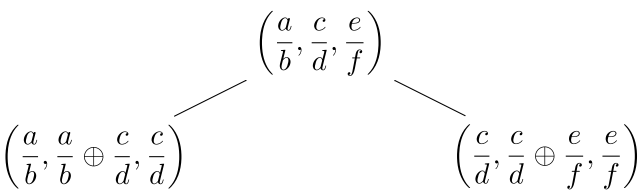
(2)

$$\left|\det\left(\frac{a}{b}, \frac{c}{d}\right)\right| = \left|\det\left(\frac{c}{d}, \frac{e}{f}\right)\right| = \left|\det\left(\frac{e}{f}, \frac{a}{b}\right)\right| = 1.$$

Define the *Farey tree* $\mathrm{F}\mathbb{T}$ as follows.

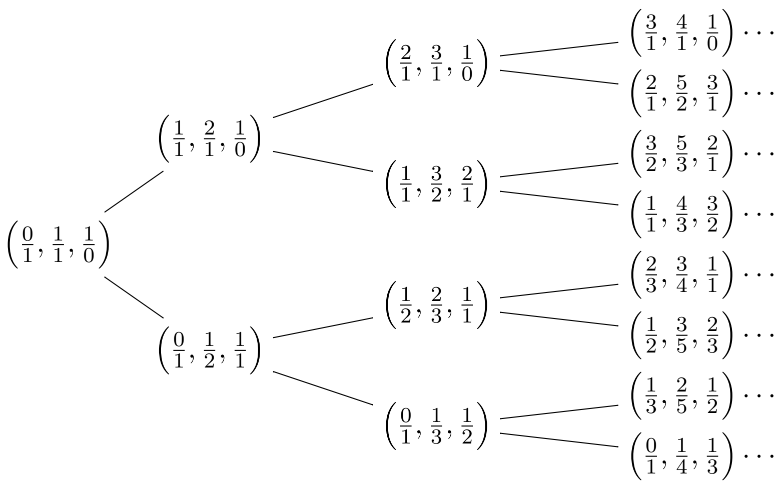
(1) The root vertex is $\left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$.

(2) Each vertex $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ has the following two children:



where $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. Here $\frac{1}{0}$ is treated as the endpoint ∞, and it is regarded as larger than every finite nonnegative fraction.

The first few vertices of $\mathrm{F}\mathbb{T}$ are as follows:



We first prove the basic properties of the Farey tree that will be used later.

PROPOSITION 5.3.2. *The following hold.*

(1) *If (r, t, s) is a Farey triple, then $(r, r \oplus t, t)$ and $(t, t \oplus s, s)$ are also Farey triples. In particular, every vertex of* $\mathrm{F}\mathbb{T}$ *is a Farey triple.*
(2) *For every reduced fraction $t \in (0, \infty)$, there exists a unique Farey triple F in* $\mathrm{F}\mathbb{T}$ *whose second component is t.*
(3) *For every (r, t, s) in* $\mathrm{F}\mathbb{T}$*, the inequalities $r < t < s$ hold.*

We first record several lemmas.

LEMMA 5.3.3. *Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$ be reduced fractions, and put $x \oplus y = \frac{a+c}{b+d}$. Then*

$$\det(x, x \oplus y) = \det(x, y), \qquad \det(x \oplus y, y) = \det(x, y).$$

In particular, if $|\det(x, y)| = 1$, then $x \oplus y$ is a reduced fraction.

PROOF. A direct computation gives

$$\det\left(\frac{a}{b}, \frac{a+c}{b+d}\right) = a(b+d) - b(a+c) = ad - bc = \det\left(\frac{a}{b}, \frac{c}{d}\right),$$

and

$$\det\left(\frac{a+c}{b+d}, \frac{c}{d}\right) = (a+c)d - (b+d)c = ad - bc = \det\left(\frac{a}{b}, \frac{c}{d}\right).$$

If $g = \gcd(a+c, b+d)$, then $g \mid ((a+c)d - (b+d)c) = ad - bc$. Thus, if $|\det(x, y)| = 1$, then $g \mid 1$, so $g = 1$. Hence $x \oplus y$ is reduced. □

LEMMA 5.3.4. *Consider the extended nonnegative rationals, consisting of nonnegative reduced fractions together with $\frac{1}{0}$. Suppose that $\frac{a}{b} < \frac{c}{d}$ and $ad - bc = -1$. Then the mediant $\frac{a+c}{b+d}$ satisfies $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Here $\frac{1}{0}$ is understood to be larger than every nonnegative rational number.*

PROOF. If $b, d > 0$, then

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc - ad}{b(b+d)} = \frac{1}{b(b+d)} > 0,$$

and

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc - ad}{d(b+d)} = \frac{1}{d(b+d)} > 0.$$

If $b = 0$ or $d = 0$, then one endpoint is $\frac{1}{0} = \infty$, and the remaining inequality follows immediately from the definition. □

LEMMA 5.3.5. *Let $(r, t, s) = \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ be a Farey triple satisfying $r < t < s$. Then $t = r \oplus s = \frac{a+e}{b+f}$.*

PROOF. Since $r < t < s$ and $|\det(r, s)| = 1$, we have $\det(r, s) = af - be = -1$. Put $\mathbf{u} = (a, b)$ and $\mathbf{w} = (e, f)$. Then $\det(\mathbf{u}^T, \mathbf{w}^T) = -1$, where det is used in the ordinary matrix sense. Hence $\mathbf{u}$ and $\mathbf{w}$ form a basis of $\mathbb{Z}^2$. Therefore $\mathbf{v} := (c, d)$ can be written uniquely as $\mathbf{v} = \alpha\mathbf{u} + \beta\mathbf{w}$. On the other hand, $r < t$ and $|\det(r, t)| = 1$ imply $\det(r, t) = ad - bc = -1$, and therefore

$$-1 = \det(\mathbf{u}^T, \mathbf{v}^T) = \det(\mathbf{u}^T, \alpha\mathbf{u}^T + \beta\mathbf{w}^T) = \beta \det(\mathbf{u}^T, \mathbf{w}^T) = \beta(-1).$$

Thus $\beta = 1$. Similarly, $t < s$ and $|\det(t, s)| = 1$ imply $\det(t, s) = cf - de = -1$, and hence

$$-1 = \det(\mathbf{v}^T, \mathbf{w}^T) = \det(\alpha\mathbf{u}^T + \mathbf{w}^T, \mathbf{w}^T) = \alpha \det(\mathbf{u}^T, \mathbf{w}^T) = \alpha(-1).$$

Thus $\alpha = 1$. Hence $\mathbf{v} = \mathbf{u} + \mathbf{w}$, that is, $(c, d) = (a+e, b+f)$. Therefore $t = \frac{c}{d} = \frac{a+e}{b+f} = r \oplus s$. □

PROOF. We prove (1). Let (r, t, s) be a Farey triple. By Lemma 5.3.3,

$$|\det(r, r \oplus t)| = |\det(r, t)| = 1,$$
$$|\det(r \oplus t, t)| = |\det(r, t)| = 1,$$

and $r \oplus t$ is reduced. Since $|\det(t, r)| = |\det(r, t)| = 1$, it follows that $(r, r \oplus t, t)$ is a Farey triple. The same argument shows that $(t, t \oplus s, s)$ is a Farey triple. Thus the property of being a Farey triple is preserved when we take children. The initial vertex $(0/1, 1/1, 1/0)$ is a Farey triple, and therefore every vertex of $\mathrm{F}\mathbb{T}$ is a Farey triple.

We next prove (3). At the initial vertex, we have $\frac{0}{1} < \frac{1}{1} < \frac{1}{0}(=\infty)$. Suppose that $r < t < s$ holds at a vertex (r, t, s). By Lemma 5.3.4, we have $r < r \oplus t < t$ and $t < t \oplus s < s$. Thus the same inequalities hold for the left child $(r, r \oplus t, t)$ and the right child $(t, t \oplus s, s)$. Hence (3) follows by induction on the depth.

We prove (2). Take a reduced fraction $t \in (0, \infty)$. Put $L_0 = \frac{0}{1}$ and $R_0 = \frac{1}{0}$. For $i \in \mathbb{Z}_{\geq 0}$, compute inductively $M_i := L_i \oplus R_i$. If $t < M_i$, put $L_{i+1} := L_i$ and $R_{i+1} := M_i$; if $t > M_i$, put $L_{i+1} := M_i$ and $R_{i+1} := R_i$. These two cases correspond to taking the left child and the right child, respectively. If $t = M_i$, stop the procedure. Writing $L_i = \frac{a_i}{b_i}$ and $R_i = \frac{c_i}{d_i}$ at each step, we have

$$M_i = \frac{a_i + c_i}{b_i + d_i},$$

and Lemma 5.3.3 shows that $|\det(L_i, R_i)| = 1$ is always preserved. In particular, L_i and R_i are always reduced. Lemma 5.3.4 also gives $L_i < t < R_i$ for all i. We first show that the procedure always stops after finitely many steps. Once both endpoints have positive denominators, the denominator $b_i + d_i$ of M_i is strictly larger than each of the two endpoint denominators b_i and d_i. The only time denominators may fail to increase monotonically is while one endpoint is $\frac{1}{0}$. In that case the procedure simply moves through the integer part: if t is an integer, it stops there, and otherwise after finitely many steps it enters an interval between two consecutive integers containing t. From then on, both endpoint denominators are positive. Now write $t = \frac{p}{q}$ in lowest terms. If both endpoints have positive denominators and $L_i = \frac{a_i}{b_i} < \frac{p}{q} < R_i = \frac{c_i}{d_i}$, then

$$b_i p - a_i q \geq 1, \qquad c_i q - p d_i \geq 1,$$

and from $b_i c_i - a_i d_i = 1$ we obtain

$$q = d_i(b_i p - a_i q) + b_i(c_i q - p d_i) \geq b_i + d_i.$$

Therefore, after both endpoint denominators become positive, the desired mediant must be reached before a mediant denominator larger than q would be required. Hence the procedure cannot continue indefinitely. Thus $t = M_n$ for some finite n. The vertex (L_n, t, R_n) is obtained by repeatedly taking children, so it is a vertex of $\mathrm{F}\mathbb{T}$. This proves existence of a vertex whose second component is t. Finally, we prove uniqueness. If a vertex (r, t, s) with second component t is given, Lemma 5.3.5 implies that $t = r \oplus s$, and (3) gives $r < t < s$. Thus the sequence of left and right choices from the initial vertex to this vertex is determined uniquely by t. If two distinct vertices had the same second component t, then there would be two different paths from the initial vertex, a contradiction. Hence such a vertex is unique. □

The Farey tree enumerates reduced fractions and is closely related to Farey sequences and the Stern–Brocot tree. It has many interesting properties beyond its connection with GM numbers, but we do not discuss them here.

Comparing the generation rules of the Farey tree and the GM tree, one sees that the rotations of components are the same. This suggests a correspondence from positive reduced fractions to GM numbers.

Definition 5.3.6. Let $f\colon \mathrm{F}\mathbb{T} \to \mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$ be an ordered complete binary tree isomorphism. Then the generation rules of the two trees induce a bijection from nonnegative reduced fractions, including the endpoints $0/1$ and $1/0$, to GM pairs. We write this correspondence as $t \mapsto (m_t, i_t)$ and call (m_t, i_t) the *fraction label* of the GM pair. Here $0/1$ and $1/0$ are treated as the left and right endpoints of the root $\left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$ of the Farey tree.

Strictly speaking, m_t and i_t also depend on k_1, k_2, k_3 and σ. Since a single GM tree will always be fixed when we use fraction labels, we suppress these parameters. This correspondence does more than merely name GM pairs; its importance will become clear when generalized strongly admissible sequences are defined later.

We close this section with a simple but important corollary.

Corollary 5.3.7. *Here we agree that $1/0=\infty$ and $1/\infty=0$. Let $t\in[0,\infty]$. Let (m_t,i_t) be the (k_1,k_2,k_3)-GM pair with fraction label t in $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$, and let $(m^*_{\frac{1}{t}},i^*_{\frac{1}{t}})$ be the (k_1,k_2,k_3)-GM pair with fraction label $\frac{1}{t}$ in $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma^*)$, where σ^* is the one appearing in Theorem 5.2.10. Then $(m_t,i_t)=(m^*_{\frac{1}{t}},i^*_{\frac{1}{t}})$.*

Proof. By Theorem 5.2.10, it suffices to show that the fraction located at the mirror-symmetric position to t in the Farey tree is $1/t$. The endpoint cases $t=0$ and $t=\infty$ are immediate from the root. For positive finite t, this follows by induction on the distance from the root to the unique Farey triple whose middle component is t. The root case is $t=1$. Suppose that a vertex (r,t,s) has mirror-symmetric vertex (s^{-1},t^{-1},r^{-1}), with the convention $0^{-1}=\infty$ and $\infty^{-1}=0$. The left child of (r,t,s) has middle label $r\oplus t$, while the right child of the mirror-symmetric vertex has middle label $t^{-1}\oplus r^{-1}=(r\oplus t)^{-1}$. The right-child case is the same. □

4. Characteristic Numbers

We finish this chapter by defining numbers called characteristic numbers. We first prove the following theorem.

Theorem 5.4.1. *Fix one tree $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$, and let (m_r,m_t,m_s) be the GM triple corresponding, via fraction labels, to a Farey triple (r,t,s). Then there exists a unique integer u satisfying*

$$\begin{cases} m_r u\equiv m_s \pmod{m_t},\\ 0<u<m_t.\end{cases}\tag{5.4.1}$$

To prove this theorem, we use the following consequence of the Euclidean algorithm. The proof here uses facts about finite regular continued fractions.

Lemma 5.4.2. *Let $x,y\in\mathbb{Z}$, and suppose that at least one of them is nonzero. Then there exist $a,b\in\mathbb{Z}$ such that $ax+by=\gcd(x,y)$.*

Proof. First assume that $x,y\geq 0$. If $y=0$, then $x>0$ by assumption, and we may take $a=1,b=0$. If $x=0$, then $y>0$, and we may take $a=0,b=1$. We therefore assume $x,y>0$. Put $d=\gcd(x,y)$ and write $x=dx'$, $y=dy'$. Let $\frac{x'}{y'}=[a_0;a_1,\dots,a_n]$ be the finite regular continued-fraction expansion. With the convention $p_{-1}=1,q_{-1}=0$ when $n=0$, put $\frac{p_i}{q_i}=[a_0;a_1,\dots,a_i]$. By Lemma 2.2.5, $x'q_{n-1}-y'p_{n-1}=(-1)^{n+1}$. If $(-1)^{n+1}=1$, take $a=q_{n-1}$ and $b=-p_{n-1}$; if $(-1)^{n+1}=-1$, take $a=-q_{n-1}$ and $b=p_{n-1}$. Then $ax'+by'=1$. Multiplying both sides by d gives $ax+by=d$, proving the assertion in this case. If $x<0$ or $y<0$, first apply the preceding argument to $|x|$ and $|y|$, and then replace a by $-a$ if $|x|=-x$, and replace b by $-b$ if $|y|=-y$. The same conclusion follows. □

Proof. We first prove existence. By Corollary 5.1.9, we have $\gcd(m_r,m_t)=1$. Hence Lemma 5.4.2 gives integers a,b such that $m_ra+m_tb=1$.

Thus a is an inverse of m_r modulo m_t. Put $u_0:=am_s$. Then

$$\begin{aligned} m_ru_0&=m_ram_s\\ &\equiv 1\cdot m_s\equiv m_s \pmod{m_t},\end{aligned}$$

so u_0 is a solution of the congruence $m_rx\equiv m_s \pmod{m_t}$. Choose the integer u satisfying $0\leq u<m_t$ and

$$u\equiv u_0 \pmod{m_t}.$$

This u also solves the same congruence. If $u=0$, then $m_s\equiv 0 \pmod{m_t}$, that is, $m_t\mid m_s$. Since $\gcd(m_t,m_s)=1$, this implies $m_t=1$. However, for a label $t\in(0,\infty)$, the corresponding entry is a middle entry of a GM-tree vertex; at the initial vertex it is $k_{\sigma(2)}+2\geq 2$, and the same lower bound is preserved along the tree. Hence $m_t\geq 2$, a contradiction. Therefore $u\neq 0$, and we obtain a solution satisfying $0<u<m_t$.

We next prove uniqueness. Suppose that u and u' both satisfy

$$m_r u \equiv m_s \pmod{m_t}, \qquad 0 < u < m_t,$$

and

$$m_r u' \equiv m_s \pmod{m_t}, \qquad 0 < u' < m_t.$$

Subtracting the two congruences gives

$$m_r(u - u') \equiv 0 \pmod{m_t},$$

that is, $m_t \mid m_r(u-u')$. Since $\gcd(m_r, m_t) = 1$, we obtain $m_t \mid (u-u')$. Moreover, $0 < u, u' < m_t$ implies

$$-(m_t - 1) \leq u - u' \leq m_t - 1.$$

The only multiple of m_t in this interval is 0, so $u - u' = 0$. Hence $u = u'$. Therefore the required integer u_t exists and is unique. □

DEFINITION 5.4.3. For any (k_1, k_2, k_3)-GM triple (m_r, m_t, m_s) in $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$, let u_t denote the unique integer u satisfying (5.4.1). We call u_t the *characteristic number*. Although this definition gives u_t for $t \in \mathbb{Q} \cap (0, \infty)$, we extend it to $t \in \mathbb{Q} \cap [0, \infty]$ by setting $u_{\frac{0}{1}} = -k_{\sigma(1)}$ and $u_{\frac{1}{0}} = 1$.

It may seem strange to use notation depending only on t for a characteristic number defined from a GM triple. However, by Proposition 5.3.2 (2), the Farey triple (r, t, s) is uniquely determined by t. Thus u_t depends only on t, and the notation is justified. We call this the *fraction labeling of characteristic numbers*. We finish the section by recording an important property of characteristic numbers.

PROPOSITION 5.4.4. *For any reduced fraction $t \in [0, 1] \cap \mathbb{Q}$, let u_t be the characteristic number with fraction label t in $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$, and put $k_t := k_{i_t}$. Let $u^*_{\frac{1}{t}}$ be the characteristic number with fraction label $\frac{1}{t}$ in $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma^*)$, where σ^* is the one appearing in Theorem 5.2.10. Then*

$$u^*_{\frac{1}{t}} = m_t - u_t - k_t.$$

The proof of this proposition requires additional tools and will be given later.

CHAPTER 6

Fence Posets and Generalized Markov Distance

In the preceding chapter we introduced generalized Markov equations, generalized Markov trees, fraction labels, and characteristic numbers. These provide the arithmetic data needed to treat generalized Markov numbers systematically. In this chapter we translate this arithmetic picture into combinatorial and geometric language by introducing fence posets and the generalized Markov distance. The main point is that generalized Markov numbers can be understood through order ideals of posets and through intersections of curves.

We first associate a fence poset with a finite sequence of positive integers and show that the number of its order ideals is encoded by continued fractions and continued-fraction matrices. Thus the continued-fraction computations of Chapter 2 can be read as elementary combinatorics of posets. We then prove skein relations for pairs of fence posets whose overlaps cross. Finally, we assign a generalized Markov length to certain curves in the plane and define the generalized Markov distance as the minimum of this length. This gives a geometric interpretation of the generalized Markov numbers introduced in the preceding chapter.

The background of this chapter lies in cluster algebra theory. Snake graphs, introduced in the surface model of cluster algebras, provide a combinatorial model for cluster variables and are closely related to Markov-type phenomena; see [**MSW11, MSW13**]. The fence posets used here are a simpler equivalent way to record the same combinatorial information in the situations needed in this text.

The presentation is based on [**BKK24, LLRS23, Ban25**]. We specialize the cluster-algebraic construction of [**BKK24**] to the present setting and present corresponding arguments for general triples (k_1, k_2, k_3), in parallel with [**LLRS23**] and [**Ban25**], which treat, respectively, the classical case $k_1 = k_2 = k_3 = 0$ and the symmetric case $k_1 = k_2 = k_3$. The proof of the results on generalized Markov distance follows the same basic ideas as [**Ban25**], but is arranged in a form adapted to the present text.

1. Order Ideals of Fence Posets and Continued Fractions

DEFINITION 6.1.1. Let $(P, \preceq)$ be a poset. For $x, y \in P$, we write $x \lessdot y$ if

$$x \prec y \quad \text{and there is no } z \in P \text{ such that } x \prec z \prec y.$$

The relation $\lessdot$ is called the *cover relation* of $(P, \preceq)$. The *Hasse diagram* of $(P, \preceq)$ is the graph satisfying the following conditions.

- Its vertex set is P.
- For distinct vertices $x, y \in P$, there is an edge between x and y if and only if either $x \lessdot y$ or $y \lessdot x$.

The diagram is drawn so that, for every cover relation, the larger element is placed above the smaller one. We do not draw orientations on the edges.

We will use posets whose Hasse diagrams are as simple as possible in the following sense.

DEFINITION 6.1.2. A finite poset $(P, \preceq)$ is called a *fence poset* if its Hasse diagram is a graph with no branching.

In this text the vertices of the Hasse diagram of a fence poset P are labeled from left to right by $1, 2, 3, \ldots, m$. Equivalently, one may identify the underlying set of P with $\{1, \ldots, m\}$ and impose a partial order on this set. Let $S = (a_0, \ldots, a_n)$ be a finite sequence of positive integers, and put

$$s_k := \sum_{i=0}^{k} a_i \qquad (k = 0, \dots, n).$$

We define a poset

$$P_S = (\{1, 2, \dots, s_n - 1\}, \preceq)$$

as follows. For $x \in P_S$, if $s_{k-1} \le x < s_k$ with $s_{-1} := 0$, set

$$\varepsilon(x) := (-1)^k.$$

For adjacent labels $x, x+1$ with $1 \le x < s_n - 1$, define the order by the cover relations

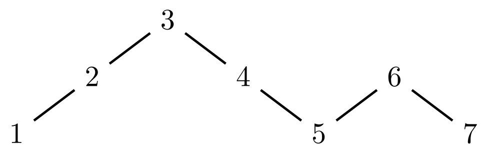
$$\begin{cases} x \lessdot x+1 & \text{if } \varepsilon(x) = 1, \\ x+1 \lessdot x & \text{if } \varepsilon(x) = -1. \end{cases}$$

Notice that the last label is $s_n - 1$, not s_n. This convention makes the numbers of edges in the successive blocks equal to

$$a_0 - 1,\ a_1,\ a_2,\ \dots,\ a_{n-1},\ a_n - 1;$$

only the first and last blocks have one fewer edge than the corresponding entry of the sequence.

Example 6.1.3. For $S = (3, 2, 1, 2)$, the Hasse diagram of P_S is



Definition 6.1.4. Let $(P, \le)$ be a poset. A subset $I \subset P$ is called an *order ideal* of $(P, \le)$ if, whenever $x \in I$, $y \in P$, and $y \le x$, one has $y \in I$. In other words, I is downward closed. We denote by $\mathcal{J}(P)$ the set of all order ideals of P.

Unlike ideals in a ring, the empty set is also regarded as an order ideal.

Example 6.1.5. The order ideals of $P_{(3,2,1,2)}$ are the following 27 subsets:

$$\begin{gathered}\emptyset, \{1\}, \{5\}, \{7\}, \{1,2\}, \{1,5\}, \{1,7\}, \{4,5\}, \{5,7\}, \{1,2,5\}, \{1,2,7\}, \{1,4,5\}, \{4,5,7\}, \{5,6,7\}, \\ \{1,2,4,5\}, \{1,2,5,7\}, \{1,4,5,7\}, \{1,5,6,7\}, \{4,5,6,7\}, \{1,2,3,4,5\}, \{1,2,4,5,7\}, \{1,2,5,6,7\}, \\ \{1,4,5,6,7\}, \{1,2,4,5,6,7\}, \{1,2,3,4,5,7\}, \{1,5,7\}, \{1,2,3,4,5,6,7\}.\end{gathered}$$

The key invariant of a fence poset in this chapter is the number of its order ideals. We write

$$N(P) := \#\mathcal{J}(P).$$

Although listing all order ideals by hand is usually cumbersome, in the case of fence posets this number is computed directly by continued fractions.

Theorem 6.1.6. *For a finite sequence of positive integers* $(a_0, a_1, \dots, a_n)$*, write*

$$N(a_0, \dots, a_n) := N(P_{(a_0, \dots, a_n)}).$$

If

$$\frac{p_n}{q_n} := [a_0; a_1, \dots, a_n]$$

is written in lowest terms, then

$$N(a_0, \dots, a_n) = p_n, \qquad N(a_1, \dots, a_n) = q_n.$$

Here we set $N() = 1$.

Proof. It suffices to prove that the numerator of $[a_0; a_1, \dots, a_n]$ is $N(a_0, \dots, a_n)$ and that the denominator is $N(a_1, \dots, a_n)$.

For $n = 0$, we have $p_0 = a_0$ and $q_0 = 1$. The poset $P_{(a_0)}$ is the chain

$$1 \prec 2 \prec \cdots \prec a_0 - 1,$$

so its order ideals are

$$\emptyset,\ \{1\},\ \{1,2\},\ \dots,\ \{1,2,\dots,a_0-1\}.$$

There are a_0 of them. Hence $N(a_0) = a_0$, and the convention $N() = 1$ gives the denominator.

For $n = 1$, the continued fraction has numerator $a_0a_1 + 1$ and denominator a_1. The poset $P_{(a_0,a_1)}$ has a unique peak at the vertex labeled a_0. Every order ideal other than the whole poset is a disjoint union of an order ideal in the left chain $\{1, \dots, a_0 - 1\}$ and an order ideal in the right chain $\{a_0 + 1, \dots, a_0 + a_1 - 1\}$. These choices give a_0a_1 ideals, and the remaining ideal is the whole poset. Thus $N(a_0, a_1) = a_0a_1 + 1$ and $N(a_1) = a_1$.

Assume now $n > 1$ and use induction on n. The standard recurrences for continued fractions show that it suffices to prove

$$\begin{aligned} N(a_0, a_1, \dots, a_n) &= a_nN(a_0, \dots, a_{n-1}) + N(a_0, \dots, a_{n-2}), \\ N(a_1, \dots, a_n) &= a_nN(a_1, \dots, a_{n-1}) + N(a_1, \dots, a_{n-2}). \end{aligned} \tag{6.1.1}$$

The second identity is proved in the same way as the first, so we prove only the first.

Suppose first that n is odd. Then $s_n - 1$ is a minimal element and s_{n-1} is a maximal element. Divide the order ideals according to whether or not they contain s_{n-1}. If an ideal contains s_{n-1}, then downward closedness forces it to contain the whole interval $\{s_{n-2}, s_{n-2} + 1, \dots, s_n - 1\}$. The remaining part is an arbitrary order ideal of the induced poset on $\{1, \dots, s_{n-2} - 1\}$, and hence gives $N(a_0, \dots, a_{n-2})$ possibilities. If an ideal does not contain s_{n-1}, then deleting the vertex s_{n-1} identifies the remaining choices with order ideals of

$$P_{(a_0,a_1,\dots,a_{n-1})} \sqcup P^*_{(a_n)},$$

where P^* denotes the dual poset. This gives

$$N(a_0, \dots, a_{n-1})N(a_n) = a_nN(a_0, \dots, a_{n-1})$$

possibilities. This proves (6.1.1) when n is odd. The even case is the same after interchanging the two cases; the term $a_nN(a_0, \dots, a_{n-1})$ comes from the ideals containing s_{n-1}, and the term $N(a_0, \dots, a_{n-2})$ from the ideals not containing it. □

Thus the number of order ideals of $P_{(a_0,\dots,a_n)}$ is the numerator of the reduced fraction $[a_0; a_1, \dots, a_n]$. Dually, the following also holds.

Theorem 6.1.7. *The statement of Theorem 6.1.6 remains true if $P_{(a_0,\dots,a_n)}$ is replaced by its dual poset $P^*_{(a_0,\dots,a_n)}$.*

Proof. The same recurrence as in the proof of Theorem 6.1.6 is obtained for the dual posets. This proves the claim. □

Corollary 6.1.8. *Let $n \geq 1$, and let $(a_0, \dots, a_n)$ be a finite sequence of positive integers. Then*

$$F_{(a_0,a_1,\dots,a_n)} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} N(a_0, \dots, a_n) & N(a_0, \dots, a_{n-1}) \\ N(a_1, \dots, a_n) & N(a_1, \dots, a_{n-1}) \end{bmatrix}.$$

Proof. This follows from Theorem 6.1.6 and the continued-fraction matrix formula, Theorem 2.2.4. □

This correspondence is the bridge between the combinatorics of fence posets and the theory of continued fractions. We will also use the following elementary consequences.

Proposition 6.1.9. *Let $(a_0, a_1, \dots, a_n)$ be a finite sequence of positive integers. Then:*

(1)
$$N(a_0, a_1, \dots, a_{n-1}, a_n) = N(a_n, a_{n-1}, \dots, a_1, a_0).$$

(2) *If* $a_n \geq 2$*, then*

$$N(a_0, a_1, \ldots, a_{n-1}, a_n) = N(a_0, a_1, \ldots, a_{n-1}, a_n - 1, 1).$$

If $a_0 \geq 2$*, then*

$$N(a_0, a_1, \ldots, a_{n-1}, a_n) = N(1, a_0 - 1, a_1, \ldots, a_{n-1}, a_n).$$

Proof. For (1), put $F_a = \left[\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Since each F_a is symmetric, we have

$$F_{a_n} F_{a_{n-1}} \cdots F_{a_0} = (F_{a_0} F_{a_1} \cdots F_{a_n})^T.$$

Comparing the $(1,1)$ entries and using Corollary 6.1.8 gives (1).

For (2), the continued-fraction identity

$$[a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_{n-1}, a_n - 1, 1]$$

for $a_n \geq 2$ gives the first equality by Theorem 6.1.6. Applying this equality to the reversed sequence and then using (1) gives the second equality. □

2. Skein Relations for Fence Posets

We next discuss relations among several fence posets. If P is a fence poset, we denote by $P(i)$ the vertex with label i. For $i \leq j$, let $P[i,j]$ be the induced subposet on the vertices with labels $i, i+1, \ldots, j$. If $i > j$, we set $P[i,j] = \emptyset$.

Definition 6.2.1. Two fence posets P_1 and P_2 are said to have an *overlap* if they contain isomorphic subposets of the form

$$R_1 = P_1[c,d], \qquad R_2 = P_2[c',d'].$$

The cases $c = d$ and $c' = d'$ are allowed. The isomorphism is required to respect the left-to-right order of the labels; in other words, the corresponding pieces must have the same shape, rather than mirror-image shapes.

An overlap is called a *crossing overlap* if the following three conditions hold:

- R_1 is upward closed in P_1, and R_2 is downward closed in P_2.
- It is not the case that $c = 1$ and $c' = 1$ simultaneously.
- It is not the case that $d = h_1$ and $d' = h_2$ simultaneously, where $h_i = |P_i|$.

Definition 6.2.2. Let P_1 and P_2 be fence posets with a crossing overlap

$$R_1 = P_1[c,d] \cong P_2[c',d'] = R_2.$$

The *type 0 resolution* of P_1 and P_2 is the four-tuple $\{P_3, P_4, P_5, P_6\}$ defined as follows.

- P_3 is obtained from $P_1[1,d] \cup P_2[d'+1, h_2]$ by keeping the induced order relations and imposing $P_1(d) \lessdot P_2(d'+1)$.
- P_4 is obtained from $P_2[1,d'] \cup P_1[d+1, h_1]$ by keeping the induced order relations and imposing $P_2(d') \gtrdot P_1(d+1)$.
- The poset P_5 is defined near the left endpoint. If $c > 1$ and $c' > 1$, then P_5 is obtained from $P_1[1, c-1] \cup P_2[1, c'-1]$ by imposing $P_1(c-1) \gtrdot P_2(c'-1)$. If $c = 1$, take the largest $v < c' - 1$ satisfying $P_2(v) \not\lessdot P_2(c'-1)$, or put $v = 0$ if no such v exists, and set $P_5 = P_2[1, v]$. If $c' = 1$, take the largest $u < c - 1$ satisfying $P_1(u) \not\gtrdot P_1(c-1)$, or put $u = 0$ if no such u exists, and set $P_5 = P_1[1, u]$.
- The poset P_6 is defined near the right endpoint. If $d < h_1$ and $d' < h_2$, then P_6 is obtained from $P_1[d+1, h_1] \cup P_2[d'+1, h_2]$ by imposing $P_1(d+1) \gtrdot P_2(d'+1)$. If $d = h_1$, take the smallest $v > d' + 1$ satisfying $P_2(v) \not\lessdot P_2(d'+1)$, or put $v = h_2 + 1$ if no such v exists, and set $P_6 = P_2[v, h_2]$. If $d' = h_2$, take the smallest $u > d + 1$ satisfying $P_1(u) \not\gtrdot P_1(d+1)$, or put $u = h_1 + 1$ if no such u exists, and set $P_6 = P_1[u, h_1]$.

Theorem 6.2.3. *Let* P_1 *and* P_2 *be fence posets with a crossing overlap. Let* $\{P_3, P_4, P_5, P_6\}$ *be their type 0 resolution. Then*

$$N(P_1)N(P_2) = N(P_3)N(P_4) + N(P_5)N(P_6). \tag{6.2.1}$$

To prove the theorem, we introduce switching points.

Definition 6.2.4. Suppose that P_1 and P_2 have an overlap $P_1[c,d] \cong P_2[c',d']$, and put $d-c=m$. For a pair of order ideals $(I_1,I_2) \in \mathcal{J}(P_1)\times\mathcal{J}(P_2)$, define the *switching point* by

$$\kappa(I_1,I_2) := \min\Big\{ i \in \{0,\dots,m\} \mid P_1(c+i)\in I_1 \iff P_2(c'+i)\in I_2 \Big\}.$$

If no such i exists, we say that the switching point does not exist.

Lemma 6.2.5. *Let P_1 and P_2 be isomorphic fence posets. For $I_1 \in \mathcal{J}(P_1)$ and $I_2 \in \mathcal{J}(P_2)$, the switching point of I_1 and I_2 does not exist if and only if*

$$(I_1,I_2)=(P_1,\emptyset) \quad \text{or} \quad (I_1,I_2)=(\emptyset,P_2).$$

Proof. Assume first that the switching point does not exist and that $P_1(1)\in I_1$ while $P_2(1)\notin I_2$. The ideal condition and the fact that the two posets have the same local cover relations force

$$P_1(2)\in I_1, \qquad P_2(2)\notin I_2.$$

Repeating the same argument along the fence gives $P_1(i)\in I_1$ and $P_2(i)\notin I_2$ for all i. Hence $I_1=P_1$ and $I_2=\emptyset$. The case $P_1(1)\notin I_1$ and $P_2(1)\in I_2$ is identical after interchanging the roles of the two posets. The converse is immediate. □

Proof of Theorem 6.2.3. It suffices to construct a bijection

$$\mathcal{J}(P_1)\times\mathcal{J}(P_2) \longrightarrow (\mathcal{J}(P_3)\times\mathcal{J}(P_4)) \sqcup (\mathcal{J}(P_5)\times\mathcal{J}(P_6)),$$

because the desired identity follows by comparing the cardinalities of the two sides. We divide the proof into two cases.

Case 1: $c>1$, $c'>1$, $d<h_1$, and $d'<h_2$. We decompose

$$\mathcal{J}(P_1)\times\mathcal{J}(P_2) = A \sqcup B$$

and construct bijections from A to $\mathcal{J}(P_3)\times\mathcal{J}(P_4)$ and from B to $\mathcal{J}(P_5)\times\mathcal{J}(P_6)$, respectively. We put $A=A_1\sqcup A_2\sqcup A_3\sqcup A_4$, where

$$A_1 := \{(I_1,I_2)\in\mathcal{J}(P_1)\times\mathcal{J}(P_2) \mid \text{a switching point exists for } I_1\cap R_1 \text{ and } I_2\cap R_2 \ \},$$

$$A_2 := \{(I_1,I_2)\in\mathcal{J}(P_1)\times\mathcal{J}(P_2) \mid I_1\cap R_1=\emptyset, \quad I_2\cap R_2=R_2 \ \},$$

$$A_3 := \left\{ (I_1,I_2)\in\mathcal{J}(P_1)\times\mathcal{J}(P_2) \;\middle|\; \begin{array}{l} I_1\cap R_1=R_1, \quad I_2\cap R_2=\emptyset, \\ P_1(d+1)\in I_1, \quad P_2(d'+1)\notin I_2 \end{array} \right\},$$

$$A_4 := \left\{ (I_1,I_2)\in\mathcal{J}(P_1)\times\mathcal{J}(P_2) \;\middle|\; \begin{array}{l} I_1\cap R_1=R_1, \quad I_2\cap R_2=\emptyset, \\ P_1(c-1)\in I_1, \quad P_2(c'-1)\notin I_2, \\ P_1(d+1)\in I_1 \Rightarrow P_2(d'+1)\in I_2 \end{array} \right\},$$

$$B := \left\{ (I_1,I_2)\in\mathcal{J}(P_1)\times\mathcal{J}(P_2) \;\middle|\; \begin{array}{l} I_1\cap R_1=R_1, \quad I_2\cap R_2=\emptyset, \\ P_1(c-1)\in I_1 \Rightarrow P_2(c'-1)\in I_2, \\ P_1(d+1)\in I_1 \Rightarrow P_2(d'+1)\in I_2 \end{array} \right\}.$$

These subsets exhaust all of $\mathcal{J}(P_1)\times\mathcal{J}(P_2)$. Indeed, by Lemma 6.2.5, if no switching point exists, then the restrictions to the overlap are necessarily either $I_1\cap R_1=R_1$, $I_2\cap R_2=\emptyset$, or $I_1\cap R_1=\emptyset$, $I_2\cap R_2=R_2$; the four subsets $A_2\sqcup A_3\sqcup A_4\sqcup B$ cover all non-switching cases.

We first define

$$\Phi_1\colon A \longrightarrow \mathcal{J}(P_3)\times\mathcal{J}(P_4).$$

The posets P_3 and P_4 are obtained by gluing subposets of P_1 and P_2. Hence every element of P_3 or P_4 is naturally identified with an element of P_1 or P_2. We denote by $P_3(i,j)$ and $P_4(i,j)$ the elements of P_3 and P_4 corresponding to the element with label j in P_i. On the crossing overlap, the same vertex may have both descriptions, one with $i=1$ and one with $i=2$.

Using this correspondence, for a pair $(I_1,I_2)\in A$ we define a pair $(I_3,I_4)\in\mathcal{J}(P_3)\times\mathcal{J}(P_4)$ as follows. Let $R_1\subset P_1$ and $R_2\subset P_2$ be the crossing overlap in P_1 and P_2, and write $R_3\subset P_3$ and $R_4\subset P_4$ for the corresponding parts in P_3 and P_4. First define the outside parts

$$I_3^{\mathrm{out}} \subset P_3\setminus R_3, \qquad I_4^{\mathrm{out}} \subset P_4\setminus R_4$$

by

$$P_3(i,j) \in I_3^{\mathrm{out}} \iff P_3(i,j) \in P_3 \setminus R_3 \text{ and, if } i=1,\ P_1(j) \in I_1, \text{ while if } i=2,\ P_2(j) \in I_2,$$
$$P_4(i,j) \in I_4^{\mathrm{out}} \iff P_4(i,j) \in P_4 \setminus R_4 \text{ and, if } i=1,\ P_1(j) \in I_1, \text{ while if } i=2,\ P_2(j) \in I_2.$$

It remains to specify only the memberships on the overlap, $I_3 \cap R_3$ and $I_4 \cap R_4$, and then we set

$$I_3 := I_3^{\mathrm{out}} \cup (I_3 \cap R_3), \qquad I_4 := I_4^{\mathrm{out}} \cup (I_4 \cap R_4).$$

Suppose first that $(I_1, I_2) \in A_1$, and let k be the label of the switching point on the P_1 side and k' the corresponding label on the P_2 side. On R_3 and R_4, membership is defined by exchanging the P_1-part and the P_2-part at the switching point. More precisely,

$$\begin{aligned} P_3(1,j) \in I_3 &\iff P_1(j) \in I_1 && (1 \le j \le k-1),\\ P_3(2,j) \in I_3 &\iff P_2(j) \in I_2 && (k' \le j \le h_2), \end{aligned}$$

and

$$\begin{aligned} P_4(2,j) \in I_4 &\iff P_2(j) \in I_2 && (1 \le j \le k'-1),\\ P_4(1,j) \in I_4 &\iff P_1(j) \in I_1 && (k \le j \le h_1). \end{aligned}$$

This definition preserves downward closedness near the switching point. Away from the overlap the ideals inherit downward closedness from I_1 and I_2. Thus I_3 and I_4 are order ideals.

For A_2, A_3, and A_4, the outside parts are again I_3^{out} and I_4^{out}, and only the memberships on the overlap are prescribed as follows:

- If $(I_1, I_2) \in A_2$, set $I_3 \cap R_3 = R_3$ and $I_4 \cap R_4 = \emptyset$.
- If $(I_1, I_2) \in A_3$, set $I_3 \cap R_3 = \emptyset$ and $I_4 \cap R_4 = R_4$.
- If $(I_1, I_2) \in A_4$, set $I_3 \cap R_3 = R_3$ and $I_4 \cap R_4 = \emptyset$.

This constructs $\Phi_1(I_1, I_2) = (I_3, I_4)$ for every $(I_1, I_2) \in A = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$.

Next we define

$$\Phi_2 \colon B \longrightarrow \mathcal{J}(P_5) \times \mathcal{J}(P_6).$$

Since $P_5 \cup P_6$ corresponds to the vertices obtained from $P_1 \cup P_2$ by deleting R_1 and R_2, for $(I_1, I_2) \in B$ we define I_5 to be the set of elements of P_5 corresponding to

$$\big(I_1 \setminus (I_1 \cap R_1)\big) \cup \big(I_2 \setminus (I_2 \cap R_2)\big),$$

and define I_6 similarly for the elements lying in P_6. By the definition of P_5 and P_6 and by the conditions defining B, the subsets I_5 and I_6 are order ideals. We set $\Phi_2(I_1, I_2) = (I_5, I_6)$. Thus we have defined a map

$$\Phi := \Phi_1 \sqcup \Phi_2$$

from $A \sqcup B$ to $\big(\mathcal{J}(P_3) \times \mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5) \times \mathcal{J}(P_6)\big)$. We prove that Φ is bijective by constructing its inverse.

Let $\langle x \rangle$ denote the order ideal generated by x. We first define

$$\Psi_1 \colon \mathcal{J}(P_3) \times \mathcal{J}(P_4) \longrightarrow \mathcal{J}(P_1) \times \mathcal{J}(P_2).$$

We decompose $\mathcal{J}(P_3) \times \mathcal{J}(P_4)$ into the following subsets A_1', A_2', A_3', and A_4':

$$A_1' := \left\{ (I_3, I_4) \in \mathcal{J}(P_3) \times \mathcal{J}(P_4) \;\middle|\; \text{a switching point exists between } I_3 \cap R_3 \text{ and } I_4 \cap R_4 \right\},$$
$$A_2' := \left\{ (I_3, I_4) \in \mathcal{J}(P_3) \times \mathcal{J}(P_4) \;\middle|\; \begin{array}{l} I_3 \cap R_3 = R_3, \quad I_4 \cap R_4 = \emptyset, \\ P_4(1, d+1) \in I_4, \quad P_3(2, d'+1) \notin I_3 \end{array} \right\},$$
$$A_3' := \left\{ (I_3, I_4) \in \mathcal{J}(P_3) \times \mathcal{J}(P_4) \;\middle|\; I_3 \cap R_3 = \emptyset, \quad I_4 \cap R_4 = R_4 \right\},$$
$$A_4' := \left\{ (I_3, I_4) \in \mathcal{J}(P_3) \times \mathcal{J}(P_4) \;\middle|\; \begin{array}{l} I_3 \cap R_3 = R_3, \quad R_4 \cap I_4 = \emptyset, \\ P_4(1, d+1) \in I_4 \Rightarrow P_3(2, d'+1) \in I_3 \end{array} \right\}.$$

The posets P_3 and P_4 are obtained by gluing subposets of P_1 and P_2, so every element of $P_1 \setminus R_1$ and $P_2 \setminus R_2$ naturally corresponds to an element of either $P_3 \setminus R_3$ or $P_4 \setminus R_4$. For any pair $(I_3, I_4) \in \mathcal{J}(P_3) \times \mathcal{J}(P_4)$, define the outside parts

$$I_1^{\mathrm{out}} \subset P_1 \setminus R_1, \qquad I_2^{\mathrm{out}} \subset P_2 \setminus R_2$$

as follows. An element $x \in P_1 \setminus R_1$ belongs to I_1^{out} precisely when the corresponding element of P_3 or P_4 belongs to I_3 or I_4, respectively. Similarly, an element $y \in P_2 \setminus R_2$ belongs to I_2^{out} precisely when the corresponding element of P_3 or P_4 belongs to I_3 or I_4, respectively. It remains to prescribe the memberships on R_1 and R_2, and then we set

$$I_1 := I_1^{\mathrm{out}} \cup (I_1 \cap R_1), \qquad I_2 := I_2^{\mathrm{out}} \cup (I_2 \cap R_2).$$

We prescribe the overlap according to the subset A'_i containing (I_3, I_4).

- If $(I_3, I_4) \in A'_1$, let k be the label of the first switching point on the P_3 side and k' the corresponding label on the P_4 side. Define membership on R_1 and R_2 by

$$\begin{aligned}
P_1(j) \in I_1 &\iff P_3(1, j) \in I_3 && (1 \le j \le k-1),\\
P_1(j) \in I_1 &\iff P_4(1, j) \in I_4 && (k \le j \le h_1),\\
P_2(j) \in I_2 &\iff P_4(2, j) \in I_4 && (1 \le j \le k'-1),\\
P_2(j) \in I_2 &\iff P_3(2, j) \in I_3 && (k' \le j \le h_2).
\end{aligned}$$

- If $(I_3, I_4) \in A'_2$, set $I_1 \cap R_1 = \emptyset$ and $I_2 \cap R_2 = R_2$.
- If $(I_3, I_4) \in A'_3$, set $I_1 \cap R_1 = R_1$ and $I_2 \cap R_2 = \emptyset$.
- If $(I_3, I_4) \in A'_4$, set $I_1 \cap R_1 = R_1$ and $I_2 \cap R_2 = \emptyset$.

With these definitions, $\Psi_1(I_3, I_4) = (I_1, I_2)$ satisfies $\Psi_1(A'_i) \subset A_i$ for every i; this is checked directly from the definitions. Hence Ψ_1 is a well-defined map from $\mathcal{J}(P_3) \times \mathcal{J}(P_4)$ to A.

We next define

$$\Psi_2 \colon \mathcal{J}(P_5) \times \mathcal{J}(P_6) \longrightarrow B.$$

The union $P_5 \cup P_6$ naturally corresponds to the set obtained from $P_1 \cup P_2$ by deleting the vertices of R_1 and R_2. Thus every element of $P_5 \cup P_6$ naturally corresponds to an element of $(P_1 \setminus R_1) \cup (P_2 \setminus R_2)$. For $(I_5, I_6) \in \mathcal{J}(P_5) \times \mathcal{J}(P_6)$, put

$$S := I_5 \cup I_6 \subset P_5 \cup P_6.$$

Using the natural correspondence above, we regard S as a subset of $(P_1 \setminus R_1) \cup (P_2 \setminus R_2)$ and define

$$I_1 := \big(S \cap (P_1 \setminus R_1)\big) \cup R_1, \qquad I_2 := S \cap (P_2 \setminus R_2).$$

Here we add R_1 because the definition of B requires $R_1 \subset I_1$. Then $(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2)$, and we obtain a map

$$\Psi_2 \colon \mathcal{J}(P_5) \times \mathcal{J}(P_6) \longrightarrow B.$$

Therefore

$$\Psi := \Psi_1 \sqcup \Psi_2 \colon \big(\mathcal{J}(P_3) \times \mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5) \times \mathcal{J}(P_6)\big) \longrightarrow \mathcal{J}(P_1) \times \mathcal{J}(P_2)$$

is defined. Tracing the correspondences shows that Ψ is the inverse of Φ. Thus Φ is bijective, and the theorem follows.

Case 2: the remaining endpoint cases. The proof is the same as in Case 1, except that the definitions of $A_1, A_2, A_3, A_4, B, A'_1, A'_2, A'_3, A'_4$ are modified at the endpoints in the natural way:

- If $c = 1$, then the condition "$P_1(c-1) \in I_1$" is omitted from the definition of A_4, and the condition "$P_1(c-1) \in I_1 \Rightarrow P_2(c'-1) \in I_2$" in the definition of B is replaced by "$P_2(c'-1) \in I_2$".
- If $c' = 1$, then the condition "$P_2(c'-1) \notin I_2$" is omitted from the definition of A_4, and the condition "$P_1(c-1) \in I_1 \Rightarrow P_2(c'-1) \in I_2$" in the definition of B is replaced by "$P_1(c-1) \notin I_1$".
- If $d = h_1$, then the condition "$P_1(d+1) \in I_1$" is omitted from the definition of A_3, and the condition "$P_1(d+1) \in I_1 \Rightarrow P_2(d'+1) \in I_2$" in the definitions of A_4 and B is replaced by "$P_2(d'+1) \in I_2$". The condition "$P_4(1, d+1) \in I_4$" is omitted from the definition of A'_2, and the condition "$P_4(1, d+1) \in I_4 \Rightarrow P_3(2, d'+1) \in I_3$" in the definition of A'_4 is replaced by "$P_3(2, d'+1) \in I_3$".

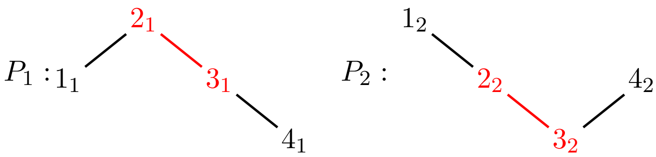
- If $d' = h_2$, then the condition "$P_2(d'+1) \notin I_2$" is omitted from the definition of A_3, and the condition "$P_1(d+1) \in I_1 \Rightarrow P_2(d'+1) \in I_2$" in the definitions of A_4 and B is replaced by "$P_1(d+1) \notin I_1$". The condition "$P_3(2, d'+1) \notin I_3$" is omitted from the definition of A_2', and the condition "$P_4(1, d+1) \in I_4 \Rightarrow P_3(2, d'+1) \in I_3$" in the definition of A_4' is replaced by "$P_4(1, d+1) \notin I_4$".

If an empty interval appears at an endpoint, it is interpreted according to the convention $P[i,j] = \emptyset$. Under this convention the maps and inverse maps constructed above still preserve the order-ideal condition in the boundary cases, and the same verification as in Case 1 shows that they are mutually inverse. This completes the proof. □

EXAMPLE 6.2.6. Let $P_1 := P_{(2,2,1)}$ and $P_2 := P_{(1,2,2)}$, and consider

$$R_1 := P_1[2,3], \qquad R_2 := P_2[2,3].$$

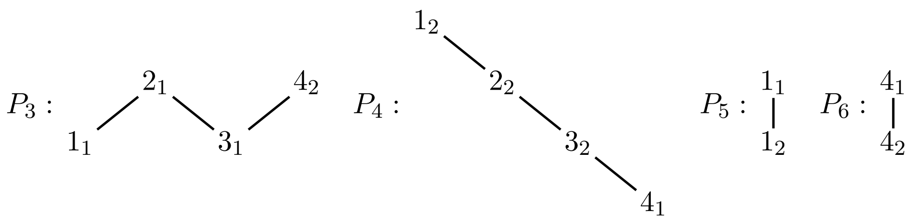
The Hasse diagrams of P_1 and P_2 are as follows; the overlap is shown in red.



Both R_1 and R_2 are two-element chains. Moreover, R_1 is upward closed in P_1 and R_2 is downward closed in P_2, so this is a crossing overlap. Hence the type 0 resolution $\{P_3, P_4, P_5, P_6\}$ is

P_3 : the poset on $P_1[1,3] \cup P_2[4,4]$ with the additional relation $P_1(3) \lessdot P_2(4)$,
P_4 : the poset on $P_2[1,3] \cup P_1[4,4]$ with the additional relation $P_2(3) \gtrdot P_1(4)$,
P_5 : the poset on $P_1[1,1] \cup P_2[1,1]$ with the additional relation $P_1(1) \gtrdot P_2(1)$,
P_6 : the poset on $P_1[4,4] \cup P_2[4,4]$ with the additional relation $P_1(4) \gtrdot P_2(4)$.

Therefore the Hasse diagrams are as follows.



The corresponding numbers of order ideals are

$$N(P_1) = N(2,2,1) = 7, N(P_2) = N(1,2,2) = 7, N(P_3) = N(2,1,2) = 8,$$
$$N(P_4) = 5, N(P_5) = N(P_6) = 3.$$

Thus

$$N(P_1)N(P_2) = 7 \cdot 7 = 49 = 8 \cdot 5 + 3 \cdot 3 = N(P_3)N(P_4) + N(P_5)N(P_6),$$

which verifies Theorem 6.2.3 in this example.

There are two endpoint variants of the same relation. We record them because they will be used for curves.

DEFINITION 6.2.7. Let two fence posets P_1, P_2 be given, and choose an integer $1 \leq i < h_2 := |P_2|$ satisfying $P_2(i) \gtrdot P_2(i+1)$. The *type 1 resolution* of P_1 and P_2 with respect to i is defined as the quadruple of fence posets $\{P_3, P_4, P_5, P_6\}$, where P_3, P_4, P_5, P_6 are as follows.

- P_3 is the poset on $P_1 \cup P_2[1,i]$ with all induced order relations, together with the additional relation $P_2(i) \lessdot P_1(1)$.
- Let $v > i$ be the smallest integer satisfying $P_2(i) \not\gtrdot P_2(v)$, if such an integer exists, and put $v = h_2 + 1$ otherwise. Let P_4 be the subposet $P_2[v, h_2]$ of P_2.
- P_5 is the poset on $P_1 \cup P_2[i+1, h_2]$ with all induced order relations, together with the additional relation $P_2(i+1) \gtrdot P_1(1)$.
- Let $u < i$ be the largest integer satisfying $P_2(i) \not\lessdot P_2(u)$, if such an integer exists, and put $u = 0$ otherwise. Let P_6 be the subposet $P_2[1,u]$ of P_2.

Theorem 6.2.8. *For a type 1 resolution of fence posets,*

$$N(P_1)N(P_2) = N(P_3)N(P_4) + N(P_5)N(P_6). \tag{6.2.2}$$

Proof. It suffices to construct a bijection

$$\mathcal{J}(P_1) \times \mathcal{J}(P_2) \longrightarrow \big(\mathcal{J}(P_3) \times \mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5) \times \mathcal{J}(P_6)\big),$$

because the desired identity follows by comparing cardinalities. We first decompose

$$\mathcal{J}(P_1) \times \mathcal{J}(P_2) = A \sqcup B,$$

where

$$A = \{(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2) \mid P_2(i+1) \in I_2 \text{ and } (P_1(1) \in I_1 \Rightarrow P_2(i) \in I_2)\},$$
$$B = \{(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2) \mid P_2(i+1) \notin I_2 \text{ or } (P_1(1) \in I_1 \text{ and } P_2(i) \notin I_2)\}.$$

We define

$$\Phi_1 \colon A \longrightarrow \mathcal{J}(P_3) \times \mathcal{J}(P_4).$$

For $(I_1, I_2) \in A$, define (I_3, I_4) by

$x \in I_3 \iff$ the element of P_1 or P_2 corresponding to x belongs to I_1 or I_2, respectively,

$y \in I_4 \iff$ the element of P_2 corresponding to y belongs to I_2.

By the defining condition of A, the subsets I_3 and I_4 are order ideals of P_3 and P_4, respectively. We set $\Phi_1(I_1, I_2) = (I_3, I_4)$.

Similarly, define

$$\Phi_2 \colon B \longrightarrow \mathcal{J}(P_5) \times \mathcal{J}(P_6).$$

For $(I_1, I_2) \in B$, define (I_5, I_6) by

$x \in I_5 \iff$ the element of P_1 or P_2 corresponding to x belongs to I_1 or I_2, respectively,

$y \in I_6 \iff$ the element of P_2 corresponding to y belongs to I_2.

By the defining condition of B, the subsets I_5 and I_6 are order ideals of P_5 and P_6, respectively. We set $\Phi_2(I_1, I_2) = (I_5, I_6)$. Thus

$$\Phi := \Phi_1 \sqcup \Phi_2$$

is a map from $A \sqcup B$ to $\big(\mathcal{J}(P_3) \times \mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5) \times \mathcal{J}(P_6)\big)$. We prove that it is bijective by constructing an inverse.

For $(I_3, I_4) \in \mathcal{J}(P_3) \times \mathcal{J}(P_4)$, define $(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2)$ by

$$P_1(j) \in I_1 \iff P_3(1, j) \in I_3,$$
$$P_2(j) \in I_2 \iff P_2(j) \in \langle P_2(i+1) \rangle \text{ or } P_3(2, j) \in I_3 \text{ or } P_4(2, j) \in I_4.$$

Then I_1 and I_2 are order ideals of P_1 and P_2. This defines a map

$$\Psi_1 \colon \mathcal{J}(P_3) \times \mathcal{J}(P_4) \longrightarrow \mathcal{J}(P_1) \times \mathcal{J}(P_2).$$

From the definition, one checks that $\Psi_1\big(\mathcal{J}(P_3) \times \mathcal{J}(P_4)\big) \subset A$.

Likewise, define

$$\Psi_2 \colon \mathcal{J}(P_5) \times \mathcal{J}(P_6) \longrightarrow B.$$

For $(I_5, I_6) \in \mathcal{J}(P_5) \times \mathcal{J}(P_6)$, define $(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2)$ by

$$P_1(j) \in I_1 \iff P_5(1, j) \in I_5,$$
$$P_2(j) \in I_2 \iff (j \geq i+1 \text{ and } P_5(2, j) \in I_5) \text{ or } (j \leq i \text{ and } P_6(j) \in I_6).$$

Then I_1 and I_2 are order ideals of P_1 and P_2. This defines a map

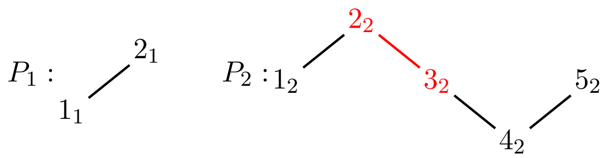
$$\Psi_2 \colon \mathcal{J}(P_5) \times \mathcal{J}(P_6) \longrightarrow \mathcal{J}(P_1) \times \mathcal{J}(P_2),$$

and from the definition one checks that $\Psi_2\big(\mathcal{J}(P_5) \times \mathcal{J}(P_6)\big) \subset B$. Thus

$$\Psi := \Psi_1 \sqcup \Psi_2 \colon \big(\mathcal{J}(P_3) \times \mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5) \times \mathcal{J}(P_6)\big) \longrightarrow A \sqcup B$$

is defined. Tracing the correspondences shows that Φ and Ψ are mutually inverse. Hence Φ is bijective, and the theorem follows. $\square$

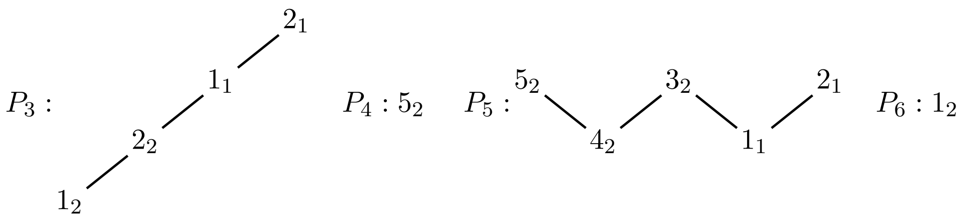
EXAMPLE 6.2.9. Let $P_1 := P_{(2,1)}$ and $P_2 := P_{(2,2,2)}$, and take $i = 2$. In P_2 we have $P_2(2) \gtrdot P_2(3)$. The diagrams are as follows; the local part used in the resolution is shown in red.



Thus we can take the type 1 resolution of P_1 and P_2 with respect to $i = 2$. In this case the smallest $v > 2$ satisfying $P_2(2) \not\gtrdot P_2(v)$ is $v = 5$, and the largest $u < 2$ satisfying $P_2(2) \not\lessdot P_2(u)$ is $u = 1$. Hence the type 1 resolution $\{P_3, P_4, P_5, P_6\}$ is

P_3 : the poset on $P_1 \cup P_2[1, 2]$ with the additional relation $P_2(2) \lessdot P_1(1)$,
$P_4 := P_2[5, 5]$,
P_5 : the poset on $P_1 \cup P_2[3, 5]$ with the additional relation $P_2(3) \gtrdot P_1(1)$,
$P_6 := P_2[1, 1]$.

Therefore the Hasse diagrams are as follows.



Hence

$$N(P_1) = N(2, 1) = 3, N(P_2) = N(2, 2, 2) = 12, N(P_3) = N(5) = 5,$$
$$N(P_4) = N(1, 1) = 2, N(P_5) = N(2, 1, 1, 2) = 13, N(P_6) = N(1, 1) = 2.$$

Therefore

$$N(P_1)N(P_2) = 3 \cdot 12 = 36 = 5 \cdot 2 + 13 \cdot 2 = N(P_3)N(P_4) + N(P_5)N(P_6),$$

so the identity (6.2.2) holds in this example.

DEFINITION 6.2.10. Let two fence posets P_1, P_2 be given. The *type 2 resolution* of P_1 and P_2 is defined as the quadruple of fence posets $\{P_3, P_4, P_5, P_6\}$, where P_3, P_4, P_5, P_6 are as follows.

- P_3 is the poset on $P_1 \cup P_2$ with all induced order relations, together with the additional relation $P_1(1) \gtrdot P_2(1)$.
- P_4 is the empty poset.
- Put $h_1 := |P_1|$. Let v be the smallest integer satisfying $v > 1$ and $P_1(1) \not\succ P_1(v)$, if such an integer exists, and put $v = h_1 + 1$ otherwise. Let P_5 be the subposet $P_1[v, h_1]$ if $v \le h_1$, and the empty poset if $v = h_1 + 1$.
- Put $h_2 := |P_2|$. Let u be the smallest integer satisfying $u > 1$ and $P_2(1) \not\prec P_2(u)$, if such an integer exists, and put $u = h_2 + 1$ otherwise. Let P_6 be the subposet $P_2[u, h_2]$ if $u \le h_2$, and the empty poset if $u = h_2 + 1$.

THEOREM 6.2.11. *For a type 2 resolution of fence posets,*

$$N(P_1)N(P_2) = N(P_3)N(P_4) + N(P_5)N(P_6). \tag{6.2.3}$$

PROOF. It suffices to construct a bijection

$$\mathcal{J}(P_1) \times \mathcal{J}(P_2) \longrightarrow (\mathcal{J}(P_3) \times \mathcal{J}(P_4)) \sqcup (\mathcal{J}(P_5) \times \mathcal{J}(P_6)),$$

because the desired identity follows by comparing cardinalities. We first decompose

$$\mathcal{J}(P_1) \times \mathcal{J}(P_2) = A \sqcup B,$$

where

$$A = \{(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2) \mid P_1(1) \in I_1 \Rightarrow P_2(1) \in I_2\},$$
$$B = \{(I_1, I_2) \in \mathcal{J}(P_1) \times \mathcal{J}(P_2) \mid P_1(1) \in I_1 \text{ and } P_2(1) \notin I_2\}.$$

We define

$$\Phi_1\colon A \longrightarrow \mathcal{J}(P_3)\times\mathcal{J}(P_4).$$

For $(I_1, I_2) \in A$, define (I_3, I_4) as follows: set $I_4 = \emptyset$, and put

$$x \in I_3 \iff \text{the element of } P_1 \text{ or } P_2 \text{ corresponding to } x \text{ belongs to } I_1 \text{ or } I_2, \text{ respectively.}$$

By the defining condition of A, the subsets I_3 and I_4 are order ideals of P_3 and P_4, respectively. We set $\Phi_1(I_1, I_2) = (I_3, I_4)$.

Similarly, define

$$\Phi_2\colon B \longrightarrow \mathcal{J}(P_5)\times\mathcal{J}(P_6).$$

For $(I_1, I_2) \in B$, define (I_5, I_6) by

$$x \in I_5 \iff \text{the element of } P_1 \text{ corresponding to } x \text{ belongs to } I_1,$$
$$y \in I_6 \iff \text{the element of } P_2 \text{ corresponding to } y \text{ belongs to } I_2.$$

By the defining condition of B, the subsets I_5 and I_6 are order ideals of P_5 and P_6, respectively. We set $\Phi_2(I_1, I_2) = (I_5, I_6)$. Thus

$$\Phi := \Phi_1 \sqcup \Phi_2$$

is a map from $A \sqcup B$ to $\big(\mathcal{J}(P_3)\times\mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5)\times\mathcal{J}(P_6)\big)$. We prove that it is bijective by constructing an inverse.

For $(I_3, I_4) \in \mathcal{J}(P_3)\times\mathcal{J}(P_4)$, define $(I_1, I_2) \in \mathcal{J}(P_1)\times\mathcal{J}(P_2)$ by

$$P_1(j) \in I_1 \iff P_3(1, j) \in I_3,$$
$$P_2(j) \in I_2 \iff P_3(2, j) \in I_3.$$

Then I_1 and I_2 are order ideals of P_1 and P_2. This defines a map

$$\Psi_1\colon \mathcal{J}(P_3)\times\mathcal{J}(P_4) \longrightarrow \mathcal{J}(P_1)\times\mathcal{J}(P_2).$$

From the definition, one checks that $\Psi_1\big(\mathcal{J}(P_3)\times\mathcal{J}(P_4)\big) \subset A$.

Likewise, define

$$\Psi_2\colon \mathcal{J}(P_5)\times\mathcal{J}(P_6) \longrightarrow B.$$

Here $\langle P_1(1)\rangle$ denotes the principal order ideal generated by $P_1(1)$, namely

$$\{x \in P_1 \mid x \preceq P_1(1)\}.$$

For $(I_5, I_6) \in \mathcal{J}(P_5)\times\mathcal{J}(P_6)$, define $(I_1, I_2) \in \mathcal{J}(P_1)\times\mathcal{J}(P_2)$ by

$$P_1(j) \in I_1 \iff P_1(j) \in \langle P_1(1)\rangle \text{ or } P_5(1, j) \in I_5,$$
$$P_2(j) \in I_2 \iff P_6(j) \in I_6.$$

Then I_1 and I_2 are order ideals of P_1 and P_2. This defines a map

$$\Psi_2\colon \mathcal{J}(P_5)\times\mathcal{J}(P_6) \longrightarrow \mathcal{J}(P_1)\times\mathcal{J}(P_2),$$

and from the definition one checks that $\Psi_2\big(\mathcal{J}(P_5)\times\mathcal{J}(P_6)\big) \subset B$. Thus

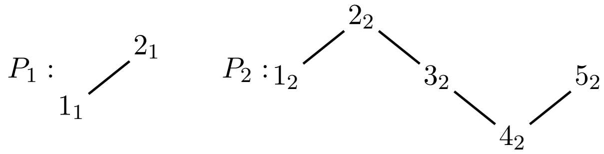
$$\Psi := \Psi_1 \sqcup \Psi_2\colon \big(\mathcal{J}(P_3)\times\mathcal{J}(P_4)\big) \sqcup \big(\mathcal{J}(P_5)\times\mathcal{J}(P_6)\big) \longrightarrow A \sqcup B$$

is defined. Tracing the correspondences shows that Φ and Ψ are mutually inverse. Hence Φ is bijective, and the theorem follows. □

EXAMPLE 6.2.12. Let $P_1 := P_{(2,1)}$ and $P_2 := P_{(2,2,2)}$, and consider

$$R_1 := P_1[2, 2], \qquad R_2 := P_2[1, 1].$$

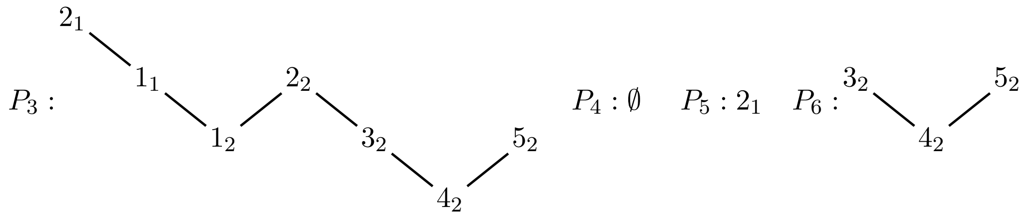
The Hasse diagrams are as follows.



Here the smallest $v > 1$ satisfying $P_1(1) \not\gtrdot P_1(v)$ is $v = 2$, and the smallest $u > 1$ satisfying $P_2(1) \not\lessdot P_2(u)$ is $u = 3$. Hence the type 2 resolution $\{P_3, P_4, P_5, P_6\}$ is

$$
\begin{aligned}
&P_3: \text{ the poset on } P_1 \cup P_2 \text{ with the additional relation } P_1(1) \gtrdot P_2(1),\\
&P_4 := \emptyset,\\
&P_5 := P_1[2,2],\\
&P_6 := P_2[3,5].
\end{aligned}
$$

Therefore the Hasse diagrams are as follows.



Consequently,

$$
\begin{aligned}
&N(P_1) = N(2,1) = 3, N(P_2) = N(2,2,2) = 12, N(P_3) = N(3,1,2,2) = 26,\\
&N(P_4) = 1, N(P_5) = N(1,1) = 2, N(P_6) = N(2,2) = 5.
\end{aligned}
$$

Thus

$$
N(P_1)N(P_2) = 3 \cdot 12 = 36 = 26 \cdot 1 + 2 \cdot 5 = N(P_3)N(P_4) + N(P_5)N(P_6),
$$

which verifies (6.2.3).

The identities in Theorems 6.2.3, 6.2.8, and 6.2.11 are called *skein relations*. The name comes from the geometric picture: two crossing curves are resolved in two possible ways, and the two products of order-ideal numbers satisfy the above exchange relation.

3. Generalized Markov Length and Generalized Markov Distance

Fix $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$. We place marked points in $\mathbb{R}^2$ as follows:

(1) every lattice point $(a, b) \in \mathbb{Z}^2$;
(2) if $k_{\sigma(1)} \neq 0$, every point of the form $(a/2, b)$ with $(a, b) \in \mathbb{Z}^2$;
(3) if $k_{\sigma(2)} \neq 0$, every point of the form $(a/2, b/2)$ with $(a, b) \in \mathbb{Z}^2$;
(4) if $k_{\sigma(3)} \neq 0$, every point of the form $(a, b/2)$ with $(a, b) \in \mathbb{Z}^2$.

We also draw all lines of slopes 0, -1, and ∞ through lattice points. These lines triangulate the plane. The resulting marked triangulated plane will be denoted by $\widetilde{\mathbb{R}^2}$. An *edge* of $\widetilde{\mathbb{R}^2}$ means a segment of one of these triangulating lines between two consecutive marked points. Such an edge need not be a full side of one of the smallest triangles, because a midpoint may split it into two edges.

A *curve segment* in $\widetilde{\mathbb{R}^2}$ is a curve whose endpoints are marked points and that satisfies the following conditions:

- its interior does not pass through a marked point;
- whenever it meets an edge of $\widetilde{\mathbb{R}^2}$, the intersection consists of one point and the curve crosses the edge transversely;
- it has only finitely many self-intersections.

In this section we mainly consider curve segments whose two endpoints are lattice points.

Definition 6.3.1. A curve segment γ in $\widetilde{\mathbb{R}^2}$ is called a *generalized arc* if its two endpoints are lattice points and if no two consecutive edge crossings of γ occur on the same edge of $\widetilde{\mathbb{R}^2}$.

We now associate a finite sign sequence with an oriented generalized arc. The signs are assigned to the triangles and edges crossed by the arc.

Definition 6.3.2. Let γ be an oriented generalized arc in $\widetilde{\mathbb{R}^2}$.

(1) Suppose that an endpoint of γ is a lattice point. Consider a triangle having that endpoint as one of its vertices and crossed by γ through the side opposite that vertex. To such a triangle we assign one sign, either $-$ or $+$; see Figure 1. If γ crosses a side that is not opposite to the endpoint, then no sign is assigned to that triangle. We call

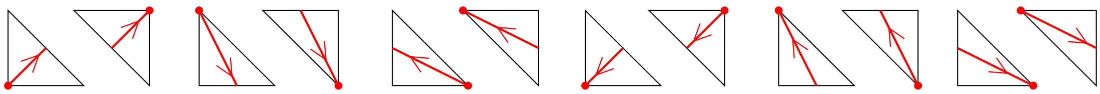


FIGURE 1. $-$ or $+$ signs assigned at an endpoint triangle

this rule the *endpoint rule* for γ. For each such endpoint triangle, the assigned sign may be chosen to be either $-$ or $+$. This arbitrariness does not affect the later theory, as noted in Remark 6.3.5.

(2) For each triangle of $\widetilde{\mathbb{R}^2}$ crossed by γ, assign a sign from $\{+,-\}$ according to the following rules.
   (i) If cutting the triangle along γ makes the region on the left-hand side of γ into a quadrilateral, assign the sign $-$ to that triangle; see Figure 2.

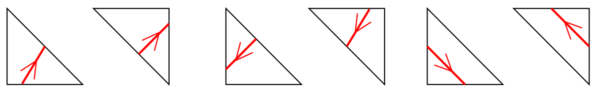


FIGURE 2. Right triangles to which the sign $-$ is assigned

   (ii) Assign the sign $+$ to every other such triangle; see Figure 3.

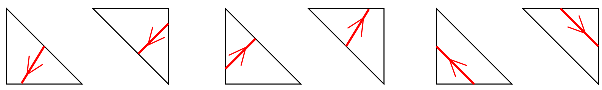


FIGURE 3. Right triangles to which the sign $+$ is assigned

We call this rule the *triangle-crossing rule* for γ.

(3) For each edge contained in a triangle whose interior meets γ, assign signs by the following rules.
   (i) For each horizontal edge, diagonal edge, or vertical edge, if its midpoint lies on the left side of γ, then assign respectively $k_{\sigma(1)}$, $k_{\sigma(2)}$, or $k_{\sigma(3)}$ copies of the sign $-$; see Figure 4.

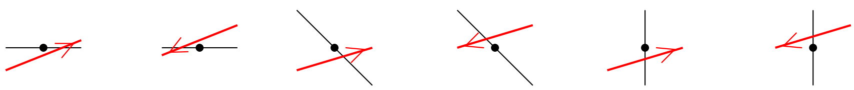


FIGURE 4. Edges to which the sign $-$ is assigned

   (ii) For each horizontal edge, diagonal edge, or vertical edge, if its midpoint lies on the right side of γ, then assign respectively $k_{\sigma(1)}$, $k_{\sigma(2)}$, or $k_{\sigma(3)}$ copies of the sign $+$; see Figure 5.
   (iii) Suppose that, after crossing one edge, the arc immediately crosses an adjacent edge without crossing the interior of a triangle; see Figures 6 and 7. In this exceptional case, first determine the sign type $+$ or $-$ for the two edges by rules (i) and (ii); the two edges receive the same sign type. If the two edges are horizontal, diagonal, or vertical, then assign respectively $k_{\sigma(1)} - 1$, $k_{\sigma(2)} - 1$, or $k_{\sigma(3)} - 1$ copies of that sign to each of the two edges, and additionally assign one sign of the opposite type to the marked point shared by the two edges. If $k_{\sigma(i)} = 0$, then the midpoint of the corresponding edge is not a point of $\widetilde{\mathbb{R}^2}$, so this situation cannot occur by the definition of a generalized arc.

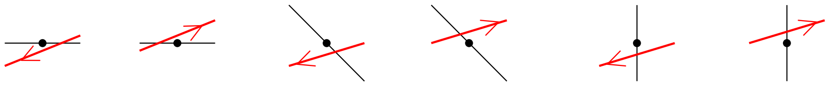


FIGURE 5. Edges to which the sign $+$ is assigned

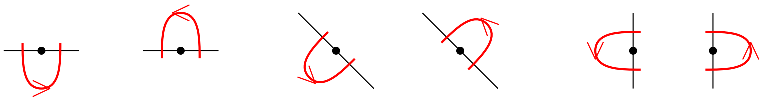


FIGURE 6. Exceptional treatment of the $-+-$ type

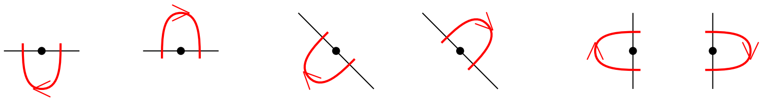


FIGURE 7. Exceptional treatment of the $+-+$ type

We call this rule the *edge-crossing rule* for γ.

Fix (k_1, k_2, k_3) and σ, and let γ be a generalized arc. We define a finite sequence $s(\gamma)$ as follows.

(1) List, in the order in which γ passes them, all signs assigned by the endpoint, triangle-crossing, and edge-crossing rules. In the exceptional edge case, the sign assigned to the common marked point is placed between the two edge contributions.
(2) Compress this sign string into the sequence of run lengths of consecutive equal signs. The resulting sequence of positive integers is denoted by

$$s(\gamma) = (a_0, \dots, a_n)$$

and is called the *sign sequence associated with* γ.

If $s(\gamma) = (a_0, \dots, a_n)$, define a fence poset P_γ as follows. Its underlying set is

$$\{1, 2, \dots, \sum_{i=0}^{n} a_i - 1\}.$$

Reading the signs from the initial endpoint of γ, impose

$$i \lessdot i+1 \quad \text{if the } (i+1)\text{-st sign is } +,$$

and

$$i \gtrdot i+1 \quad \text{if the } (i+1)\text{-st sign is } -,$$

for $1 \leq i \leq \sum_{i=0}^{n} a_i - 2$. The first and last signs do not affect the cover relations of P_γ.

The number of order ideals of P_γ is called the (k_1, k_2, k_3, σ)*-generalized Markov length*, or simply the *GM length*, of γ. It is denoted by $|\gamma|$. Although the notation does not show it, both $s(\gamma)$ and $|\gamma|$ depend on (k_1, k_2, k_3) and σ.

PROPOSITION 6.3.3. *If $s(\gamma) = (a_0, a_1, \dots, a_n)$, then P_γ is isomorphic either to $P_{(a_0,\dots,a_n)}$ or to its dual $P^*_{(a_0,\dots,a_n)}$. In particular,*

$$|\gamma| = N(a_0, \dots, a_n).$$

PROOF. The sequence $s(\gamma) = (a_0, \dots, a_n)$ records the lengths of the maximal blocks of equal signs. The poset P_γ changes the direction of its cover relations precisely when the sign changes. Thus the successive blocks of cover relations in P_γ have lengths

$$a_0 - 1,\ a_1,\ a_2, \dots,\ a_{n-1},\ a_n - 1,$$

which are exactly the block lengths in the definition of $P_{(a_0,\dots,a_n)}$. If the first relevant sign gives the same orientation as in the definition of $P_{(a_0,\dots,a_n)}$, then $P_\gamma \cong P_{(a_0,\dots,a_n)}$; otherwise all cover relations are reversed, and $P_\gamma \cong P^*_{(a_0,\dots,a_n)}$. The formula for $|\gamma|$ follows from Theorem 6.1.6 and Theorem 6.1.7. □

EXAMPLE 6.3.4. For $(k_1, k_2, k_3, \sigma) = (1, 2, 0, \mathrm{id})$, one generalized arc has sign sequence

$$s(\gamma) = (3, 7, 1, 6, 2, 1, 3, 1, 2, 2, 6, 5).$$

Therefore Theorem 6.1.6 gives

$$|\gamma| = N(3, 7, 1, 6, 2, 1, 3, 1, 2, 2, 6, 5) = 551409.$$

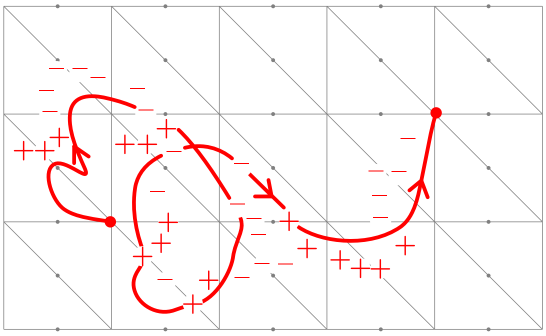


FIGURE 8. A generalized arc γ and the assigned signs

REMARK 6.3.5. By Proposition 6.1.9 (1), the value $|\gamma|$ is unchanged if the orientation of γ is reversed. By Proposition 6.1.9 (2), it is also unchanged by the arbitrary choices of endpoint signs in the endpoint rule.

We now define the distance obtained by minimizing GM length.

DEFINITION 6.3.6. For lattice points A, B in $\widetilde{\mathbb{R}^2}$, put

$$\Gamma(A, B) := \{\gamma \mid \gamma \text{ is a generalized arc joining } A \text{ and } B\}.$$

Define

$$d(A, B) := \begin{cases} \min\limits_{\gamma \in \Gamma(A,B)} |\gamma|, & A \neq B, \\ 0, & A = B. \end{cases}$$

We call $d(A, B)$ the (k_1, k_2, k_3, σ)*-generalized Markov distance*, or simply the *GM distance*, between A and B. Despite the terminology, this is not a metric in general, since it need not satisfy the triangle inequality.

The next theorem identifies the arcs that realize this minimum. Let γ^R_{AB} be a generalized arc obtained by slightly perturbing the straight segment from A to B to the right, and let γ^L_{AB} be the analogous left perturbation. The perturbation is chosen so that it crosses the same triangles and edges that the straight segment crosses, but avoids the marked points of $\widetilde{\mathbb{R}^2}$. Figure 9 illustrates these two perturbations.

THEOREM 6.3.7. *For two lattice points $A, B \in \widetilde{\mathbb{R}^2}$,*

$$d(A, B) = |\gamma^R_{AB}| = |\gamma^L_{AB}|.$$

We first record several lemmas.

LEMMA 6.3.8. *For any two lattice points A and B, one has*

$$|\gamma^R_{AB}| = |\gamma^L_{AB}|.$$

PROOF. The two arcs have sign sequences that are reverse to each other, up to the harmless endpoint modifications described in Remark 6.3.5. The assertion follows from Proposition 6.1.9. □

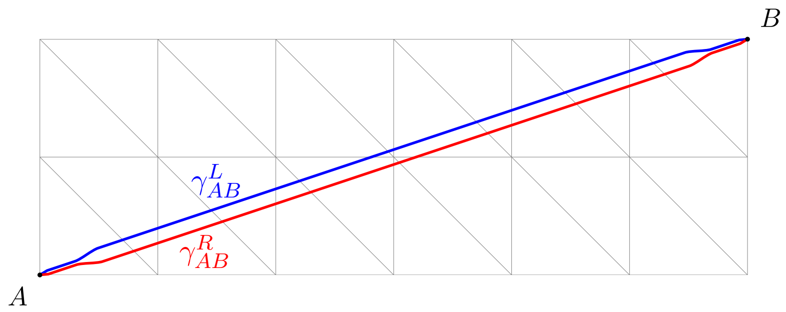


FIGURE 9. γ^R_{AB} and γ^L_{AB}

LEMMA 6.3.9. *Let A, B be lattice points and let $v \in \mathbb{Z}^2$. Then $d(A + v, B + v) = d(A, B)$.*

PROOF. Translation by v preserves the triangulation, the marked points, the signs assigned by the rules above, and hence GM length. Taking minima gives the claim. $\square$

LEMMA 6.3.10. *Let γ be a generalized arc with endpoints A and B. If γ has a non-contractible self-intersection, then*

$$|\gamma| > d(A, B).$$

PROOF. The assertion is clear when $A = B$, so assume that A and B are distinct. Let C be a self-intersection point of γ. Since γ passes through C twice, we can decompose γ into three curve pieces

$$\gamma = \gamma_1\gamma_2\gamma_3,$$

where γ_1 is the part up to the first passage through C, γ_2 is the part between the two passages through C, and γ_3 is the part after the second passage through C. With the orientations induced from γ, the curve γ_1 starts at A, and γ_3 ends at B.

Let γ' be the generalized arc obtained by concatenating γ_1 and γ_3 and then repeatedly applying the cutting-and-gluing operation of Figure 10 whenever the resulting curve passes through the same edge twice in succession. In this construction, we keep the positions of the intersections of γ_1 and γ_3 with the edges of $\widetilde{\mathbb{R}^2}$ unchanged. Among the intersection points of γ' with edges of $\widetilde{\mathbb{R}^2}$, let p_1 be the one contained in γ_1 and farthest from A along the curve, and let p_3 be the one contained in γ_3 and farthest from B. Let Δ be the triangle of $\widetilde{\mathbb{R}^2}$ containing both p_1 and p_3.

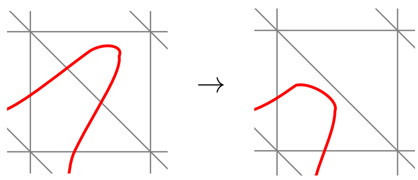


FIGURE 10. Cutting and gluing when the midpoint of the central diagonal is not a marked point of $\widetilde{\mathbb{R}^2}$

We first consider the case where Δ is not a triangle containing an endpoint of γ. Figure 11 gives an example of the local configuration of γ and γ'; the curve γ' need not cross the right-angled part exactly as in the picture, but the same argument applies in all cases.

Put $h = |P_\gamma|$. By the construction of γ', there are integers $u < v$ such that $P_{\gamma'}$ is obtained from the induced subposet on

$$P_\gamma[1, u] \cup P_\gamma[v, h]$$

by adding one edge between $P_\gamma(u)$ and $P_\gamma(v)$. In what follows we use the same integer labels for the corresponding elements of $P_{\gamma'}$.

We prove $|\gamma| > |\gamma'|$. It suffices to give an injective but non-surjective map

$$\mathcal{J}(P_{\gamma'}) \longrightarrow \mathcal{J}(P_\gamma).$$

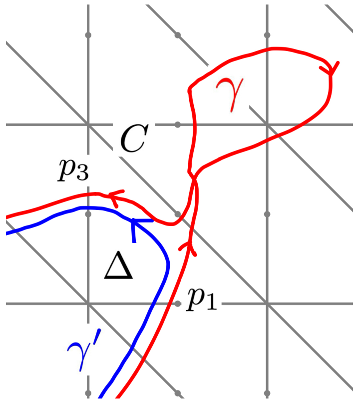


FIGURE 11. An example of a local configuration

For an order ideal $I' \in \mathcal{J}(P_{\gamma'})$, define an order ideal $I \in \mathcal{J}(P_\gamma)$ as follows. For $x \in [1, u-1] \cup [v+1, h]$, impose

$$P_{\gamma'}(x) \in I' \quad \Longleftrightarrow \quad P_\gamma(x) \in I.$$

At the two gluing endpoints we impose

$$P_{\gamma'}(u) \notin I' \Rightarrow P_\gamma(u) \notin I, \qquad P_{\gamma'}(u) \in I' \Rightarrow \langle P_\gamma(u) \rangle \subset I,$$
$$P_{\gamma'}(v) \notin I' \Rightarrow P_\gamma(v) \notin I, \qquad P_{\gamma'}(v) \in I' \Rightarrow \langle P_\gamma(v) \rangle \subset I.$$

For all remaining $x \in [u+1, v-1]$, put $P_\gamma(x) \notin I$. This gives an injective map

$$\Phi : \mathcal{J}(P_{\gamma'}) \longrightarrow \mathcal{J}(P_\gamma).$$

We show that Φ is not surjective. Suppose first that $u \notin \langle P_\gamma(v) \rangle$ or $v \notin \langle P_\gamma(u) \rangle$. Then the interval $P_\gamma[u+1, v-1]$ contains either a minimal element or a maximal element of P_γ. Let this element be $P_\gamma(w)$. Then $\langle P_\gamma(w) \rangle$ is an order ideal of P_γ, but it is not in the image of Φ.

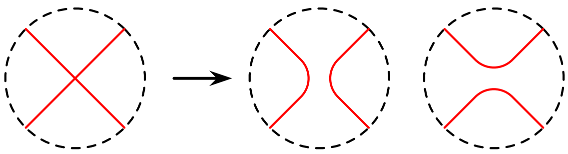
It remains to consider the case $u \in \langle P_\gamma(v) \rangle$ and $v \in \langle P_\gamma(u) \rangle$. In this case, the order ideal $\langle P_\gamma(u+1) \rangle$ is not in the image of Φ; equivalently, one may use $\langle P_\gamma(v-1) \rangle$. Hence Φ is never surjective.

The cases where Δ is the first or the last triangle are treated by the same construction of Φ, with the evident endpoint modification, and the same failure of surjectivity. Therefore

$$|\gamma| > |\gamma'| \geq d(A, B),$$

which proves the desired inequality. □

We now use the idea of *skein resolution*. This is the standard operation of replacing a crossing of curves by two pairs of locally smoothed curve pieces. Locally the operation is represented as follows.



In the present text we apply this operation to intersections of two distinct generalized arcs, not to self-intersections. Resolving one crossing produces four generalized arcs in total, two from one smoothing and two from the other. If a smoothed curve is not itself a generalized arc because it passes through the same edge of $\widetilde{\mathbb{R}^2}$ twice in succession, we replace it by the homotopic generalized arc obtained by the cutting-and-gluing operation of Figure 10.

THEOREM 6.3.11. *Let γ_1 and γ_2 be generalized arcs in $\widetilde{\mathbb{R}^2}$ that intersect in minimal position. Fix one intersection point C, and let $\gamma_3, \gamma_4, \gamma_5, \gamma_6$ be the four generalized arcs obtained by skein resolving this crossing. Here γ_3 starts at the initial point of γ_1 and ends at the terminal point of γ_2, γ_4 starts at the initial point of γ_2 and ends at the terminal point of γ_1, γ_5 starts at the initial* *point of γ_1 and ends at the initial point of γ_2, and γ_6 starts at the terminal point of γ_1 and ends at the terminal point of γ_2. Then, for every $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and every $\sigma \in \mathfrak{S}_3$,*

$$|\gamma_1|\,|\gamma_2| = |\gamma_3|\,|\gamma_4| + |\gamma_5|\,|\gamma_6|.$$

PROOF. Consider the triangle or edge of $\widetilde{\mathbb{R}^2}$ containing the chosen crossing C. There are two possibilities. The crossing may be movable, in the sense that by a homotopy one can move it between adjacent triangles and edges, or it may be non-movable. In the latter case the crossing necessarily lies in the interior of a triangle; see Figure 12.

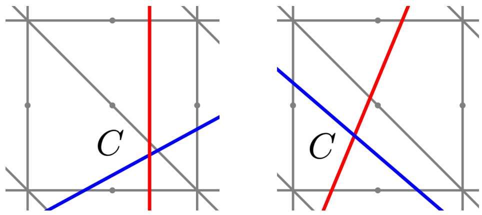


FIGURE 12. A movable crossing point (left) and a non-movable crossing point (right)

First assume that the crossing is movable. Then the two arcs contain a common parallel-running interval, made of triangles and edges through which the two arcs pass in the same way, and containing the crossing C. Orient γ_1 and γ_2 so that they run through this common interval in the same direction, and consider the associated fence posets P_{γ_1} and P_{γ_2}. Let $P_{\gamma_1}[c, d]$ and $P_{\gamma_2}[c', d']$ be the subposets determined by the signs in the whole parallel-running interval. Then

$$P_{\gamma_1}[c, d] \cong P_{\gamma_2}[c', d'].$$

Moreover the inequality immediately before the interval in P_{γ_1} is reversed relative to the corresponding inequality in P_{γ_2}, and the same is true at the end of the interval. Otherwise the parallel interval could be extended. The left-right relation of γ_1 and γ_2 must also be reversed between the entrance and exit of the interval; otherwise the two arcs would not cross in that interval. Hence P_{γ_1} and P_{γ_2} have a crossing overlap along these subposets.

Label the curve for which the overlapping subposet is upward closed by γ_1, and the other one by γ_2. Applying the type 0 skein relation (6.2.1) to $P_1 = P_{\gamma_1}$ and $P_2 = P_{\gamma_2}$ gives the desired identity. Indeed, the two arcs obtained by reconnecting along the orientations correspond to the posets P_3 and P_4 in (6.2.1); these are precisely P_{γ_3} and P_{γ_4}. The other smoothing pair corresponds to P_5 and P_6, namely P_{γ_5} and P_{γ_6}. Proposition 6.3.3 identifies $N(P_{\gamma_i})$ with $|\gamma_i|$, so the asserted formula follows.

It remains to treat the case where C is non-movable and lies in a triangle. There are three cases:

(1) C lies in the first or last triangle of γ_1, and in an interior triangle of γ_2;
(2) C lies in the first or last triangle of both γ_1 and γ_2;
(3) neither of the preceding cases occurs.

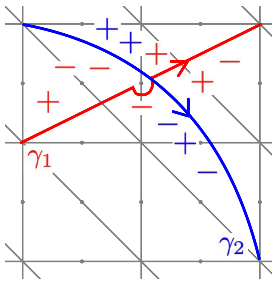
If C lies in the first or last triangle of γ_2 and in an interior triangle of γ_1, then interchanging the roles of γ_1 and γ_2 reduces this to case (1).

In case (1), orient γ_2 so that the sign assigned by the sign rule to the triangle containing C is $-$. Let $i, i+1$ be the two adjacent vertices of P_{γ_2} whose order relation is determined by this sign. Applying the type 1 resolution to P_{γ_1} and P_{γ_2} at this i, Theorem 6.2.8 gives the desired identity.

In case (2), orient both arcs so that the endpoint contained in the triangle of C is the initial point. Applying the type 2 resolution, Theorem 6.2.11 gives the desired identity.

Finally, consider case (3). Among the three sides of the triangle containing C, there is a side crossed by both γ_1 and γ_2. Let this side be ℓ. Its midpoint is necessarily a marked point of $\widetilde{\mathbb{R}^2}$; write the two edges of $\widetilde{\mathbb{R}^2}$ obtained by splitting ℓ at this midpoint as ℓ_1 and ℓ_2. The arcs γ_1 and γ_2 meet different ones of these two edges. We may assume that γ_1 meets ℓ_1 and γ_2 meets ℓ_2. Orient the arcs so that they pass through ℓ_1 or ℓ_2 before passing through C. Then the element of P_{γ_1} whose order relations are determined by the sign of ℓ_1 and the sign of the triangle containing C, and the corresponding element of P_{γ_2} determined by ℓ_2 and the same triangle, form a crossing overlap. Applying the type 0 skein relation (6.2.1) proves the formula in this last case as well. ☐

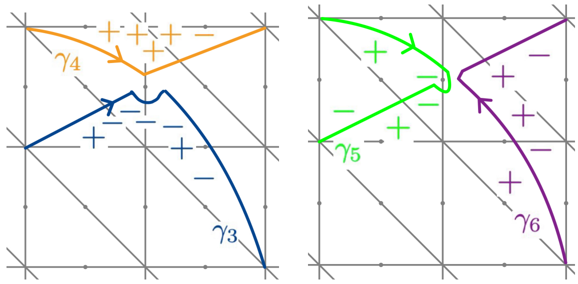
EXAMPLE 6.3.12. Let $(k_1, k_2, k_3) = (1, 1, 1)$. Since this is the symmetric case, the choice of σ is irrelevant. Consider the two arcs γ_1 and γ_2 shown below.



Then

$$|\gamma_1| = N(1, 3, 2, 1) = 13, \qquad |\gamma_2| = N(2, 1, 1, 1) = 8.$$

Resolving their crossing gives the four arcs $\gamma_3, \gamma_4, \gamma_5, \gamma_6$ below.



For these arcs one has

$$|\gamma_3| = N(1, 4, 1, 1) = 11, |\gamma_4| = N(4, 1) = 5, |\gamma_5| = N(1, 2, 1, 1) = 7, |\gamma_6| = N(1, 1, 2, 1) = 7.$$

Here the sign sequence of γ_5 includes the exceptional edge treatment. Thus

$$|\gamma_1||\gamma_2| = 13 \cdot 8 = 104, \qquad |\gamma_3||\gamma_4| + |\gamma_5||\gamma_6| = 11 \cdot 5 + 7 \cdot 7 = 104,$$

and hence

$$|\gamma_1||\gamma_2| = |\gamma_3||\gamma_4| + |\gamma_5||\gamma_6|.$$

For later use we also introduce the mixed perturbations. Let M be the midpoint of A and B. The arc γ_{AB}^{RL} follows γ_{AB}^{R} from A until just after passing M, and then follows γ_{AB}^{L} to B. Similarly, γ_{AB}^{LR} follows γ_{AB}^{L} first and γ_{AB}^{R} after the midpoint.

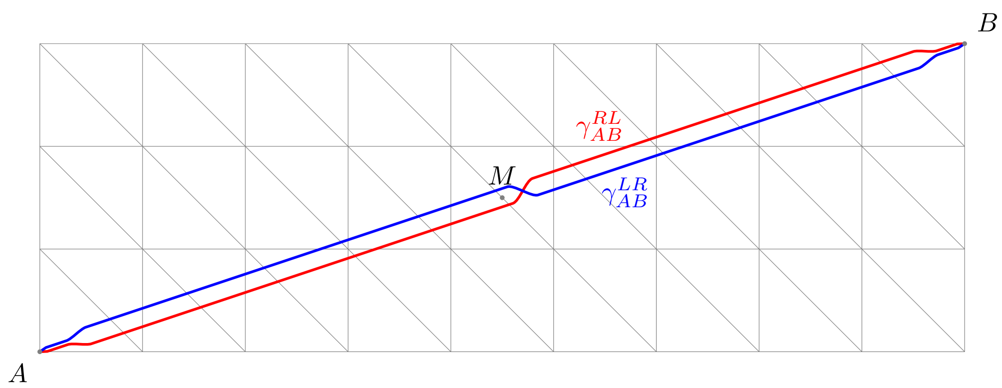


FIGURE 13. γ_{AB}^{RL} and γ_{AB}^{LR}

LEMMA 6.3.13. *Let A and B be distinct lattice points, and let M be their midpoint. If M is not a lattice point, then*

$$|\gamma_{AB}^{RL}| = |\gamma_{AB}^{LR}|.$$

PROOF. If the open segment AB contains no marked point of $\widetilde{\mathbb{R}^2}$, then the four arcs

$$\gamma_{AB}^{RL}, \quad \gamma_{AB}^{LR}, \quad \gamma_{AB}^{R}, \quad \gamma_{AB}^{L}$$

are mutually homotopic as generalized arcs. Hence their GM lengths are all equal. If the open segment AB contains exactly one marked point, then γ_{AB}^{RL} is homotopic to γ_{AB}^{R}, and γ_{AB}^{LR} is homotopic to γ_{AB}^{L}. Therefore the assertion follows from Lemma 6.3.8. In the rest of the proof we assume that the interior of AB contains at least two marked points of $\widetilde{\mathbb{R}^2}$.

Let A' be the first lattice point on the segment AB after A. We first consider the case where the second sign attached to $\gamma_{AA'}^{R}$ is $-$. Write

$$s(\gamma_{AA'}^{R}) = (a_0, \dots, a_n).$$

By choosing the endpoint signs in the triangle containing A and the triangle containing B to be $-$ and $+$, respectively, we may assume that $a_0, a_n > 1$. Then, for some $m \in \mathbb{Z}_{\geq 0}$, the sign sequences of the two arcs are

$$\begin{aligned} s(\gamma_{AB}^{RL}) = ((a_0, \dots, a_n - 1, 1, 2 + k_1 + k_2 + k_3)^m,\ a_0, \dots, a_n,& \\ (2 + k_1 + k_2 + k_3, 1, a_n - 1, \dots, a_0)^m)& \end{aligned}$$

and

$$\begin{aligned} s(\gamma_{AB}^{LR}) = ((1, a_n - 1, \dots, a_0, 2 + k_1 + k_2 + k_3)^m,\ 1, a_n - 1, \dots, a_0 - 1, 1,& \\ (2 + k_1 + k_2 + k_3, a_0, \dots, a_n - 1, 1)^m).& \end{aligned}$$

Here $(*)^m$ means that the block inside the parentheses is repeated m times. To compute $|\gamma_{AB}^{RL}|$ and $|\gamma_{AB}^{LR}|$, we consider the corresponding continued-fraction matrix products

$$\begin{aligned} S_{1,m} &:= (F_{a_0} \cdots F_{a_{n-1}} F_{a_n - 1} F_1 F_{2+k_1+k_2+k_3})^m F_{a_0} \cdots F_{a_n} (F_{2+k_1+k_2+k_3} F_1 F_{a_n - 1} F_{a_{n-1}} \cdots F_{a_0})^m, \\ S_{2,m} &:= (F_1 F_{a_n - 1} F_{a_{n-1}} \cdots F_{a_0} F_{2+k_1+k_2+k_3})^m F_1 F_{a_n - 1} F_{a_{n-1}} \cdots F_{a_1} F_{a_0 - 1} F_1 \\ &\quad \cdot (F_{2+k_1+k_2+k_3} F_{a_0} \cdots F_{a_{n-1}} F_{a_n - 1} F_1)^m, \end{aligned}$$

where $F_i = \left[\begin{smallmatrix} i & 1 \\ 1 & 0 \end{smallmatrix}\right]$. By Corollary 6.1.8 and Proposition 6.3.3, the $(1,1)$ entries of these two matrices give $|\gamma_{AB}^{RL}|$ and $|\gamma_{AB}^{LR}|$, respectively. Thus it is enough to prove $(S_{1,m})_{11} = (S_{2,m})_{11}$ for every $m \geq 0$.

Put

$$P = \begin{bmatrix} p & q \\ r & s \end{bmatrix} := F_{a_1} F_{a_2} \cdots F_{a_{n-1}}.$$

Since each F_i is symmetric, taking transposes gives

$$P^T = \begin{bmatrix} p & r \\ q & s \end{bmatrix} = F_{a_{n-1}} F_{a_{n-2}} \cdots F_{a_1}.$$

Define

$$\begin{aligned} X &:= F_{a_0} P F_{a_n - 1} F_1 F_{2+k_1+k_2+k_3}, & C &:= F_{a_0} P F_{a_n}, \\ V &:= F_{2+k_1+k_2+k_3} F_{a_0} P F_{a_n - 1} F_1, & D &:= F_1 F_{a_0 - 1} P^T F_{a_n - 1} F_1. \end{aligned}$$

Then the above products can be rewritten as

$$S_{1,m} = X^m C (X^T)^m, \qquad S_{2,m} = (V^T)^m D V^m. \tag{6.3.1}$$

Moreover, $X = F_{2+k_1+k_2+k_3}^{-1} V F_{2+k_1+k_2+k_3}$. Hence X and V have the same trace and determinant. We write

$$\tau = \operatorname{tr}(X) = \operatorname{tr}(V), \qquad \delta = \det(X) = \det(V).$$

By the Cayley–Hamilton theorem, or Corollary A.2.2, we have

$$X^2 = \tau X - \delta E_2, \qquad V^2 = \tau V - \delta E_2.$$

Repeatedly applying these identities, we see that for every $m \geq 2$ there exist integers α_m and β_m such that

$$X^m = \alpha_m X + \beta_m E_2, \qquad V^m = \alpha_m V + \beta_m E_2.$$

The same coefficients occur for X^m and V^m because the two matrices have the same trace and determinant. Taking transposes gives

$$(X^T)^m = \alpha_m X^T + \beta_m E_2, \qquad (V^T)^m = \alpha_m V^T + \beta_m E_2.$$

Substituting these expressions into (6.3.1), we obtain

$$\begin{aligned} S_{1,m} &= (\alpha_m X + \beta_m E_2) C (\alpha_m X^T + \beta_m E_2) \\ &= \alpha_m^2 X C X^T + \alpha_m \beta_m X C + \alpha_m \beta_m C X^T + \beta_m^2 C, \end{aligned}$$

and similarly

$$\begin{aligned} S_{2,m} &= (\alpha_m V^T + \beta_m E_2) D (\alpha_m V + \beta_m E_2) \\ &= \alpha_m^2 V^T D V + \alpha_m \beta_m V^T D + \alpha_m \beta_m D V + \beta_m^2 D. \end{aligned}$$

Therefore there are constants u, v, w, u', v', w' such that

$$(S_{1,m})_{11} = u\alpha_m^2 + v\alpha_m\beta_m + w\beta_m^2, \qquad (S_{2,m})_{11} = u'\alpha_m^2 + v'\alpha_m\beta_m + w'\beta_m^2. \tag{6.3.2}$$

Next we derive the recurrence satisfied by α_m and β_m. From

$$X^{m+2} = \tau X^{m+1} - \delta X^m$$

and

$$X^m = \alpha_m X + \beta_m E_2, \qquad X^{m+1} = \alpha_{m+1} X + \beta_{m+1} E_2, \qquad X^{m+2} = \alpha_{m+2} X + \beta_{m+2} E_2,$$

we obtain

$$(\alpha_{m+2} - \tau\alpha_{m+1} + \delta\alpha_m) X + (\beta_{m+2} - \tau\beta_{m+1} + \delta\beta_m) E_2 = 0.$$

Since X is a continued-fraction matrix, its $(2,1)$ entry is nonzero. Hence X and E_2 are linearly independent over $\mathbb{Q}$, and so

$$\alpha_{m+2} - \tau\alpha_{m+1} + \delta\alpha_m = 0, \qquad \beta_{m+2} - \tau\beta_{m+1} + \delta\beta_m = 0. \tag{6.3.3}$$

The characteristic equation of this recurrence is

$$t^2 - \tau t + \delta = 0.$$

Here $\delta \in \{\pm 1\}$, and since X is a continued-fraction matrix, the exceptional case $\delta = 1$ and $\tau = 2$ does not occur. Thus the characteristic equation has two distinct real roots. Let them be λ_1 and λ_2. By Theorem A.3.1, there exist real constants a, b, a', b' such that

$$\alpha_m = a\lambda_1^m + b\lambda_2^m, \qquad \beta_m = a'\lambda_1^m + b'\lambda_2^m.$$

Substituting these formulas into (6.3.2), we find constants h, i, j, h', i', j' such that

$$(S_{1,m})_{11} = h\lambda_1^{2m} + i\lambda_1^m\lambda_2^m + j\lambda_2^{2m}, \qquad (S_{2,m})_{11} = h'\lambda_1^{2m} + i'\lambda_1^m\lambda_2^m + j'\lambda_2^{2m}.$$

Therefore both sequences $(S_{1,m})_{11}$ and $(S_{2,m})_{11}$ satisfy the same linear recurrence whose characteristic polynomial is

$$(t - \lambda_1^2)(t - \lambda_1\lambda_2)(t - \lambda_2^2).$$

Expanding this polynomial gives

$$t^3 - (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)t^2 + \lambda_1\lambda_2(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)t - \lambda_1^3\lambda_2^3.$$

Since $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \delta$, this common recurrence is

$$T_{m+3} = (\tau^2 - \delta)T_{m+2} - \delta(\tau^2 - \delta)T_{m+1} + \delta^3 T_m.$$

Thus, to prove $(S_{1,m})_{11} = (S_{2,m})_{11}$ for all m, it suffices to check the equality for $m = 0, 1, 2$. These three initial equalities are verified by direct multiplication of the displayed matrices. Hence

$$|\gamma_{AB}^{RL}| = |\gamma_{AB}^{LR}|$$

when the second sign attached to $\gamma_{AA'}^{R}$ is $-$.

Finally, suppose that the second sign attached to $\gamma^R_{AA'}$ is $+$. In this case the two descriptions of $s(\gamma^{RL}_{AB})$ and $s(\gamma^{LR}_{AB})$ above are interchanged. Accordingly, the roles of the products $S_{1,m}$ and $S_{2,m}$ are interchanged, while the conjugacy relation, the common recurrence, and the verification for $m = 0, 1, 2$ are unchanged. The same argument therefore proves the desired equality in this case as well. □

LEMMA 6.3.14. *Let A and B be lattice points, and let M be their midpoint. If M is a lattice point, then*

$$|\gamma^{RL}_{AB}| = |\gamma^{LR}_{AB}|.$$

PROOF. For the same reason as in Lemma 6.3.13, we may assume that the open segment AB contains at least two lattice points. Write

$$s(\gamma^R_{AM}) = (a_1, \dots, a_n),$$

where the sign on the endpoint triangle supplied by the endpoint rule is chosen to agree with the adjacent sign.

First suppose that the second sign attached to γ^R_{AM} is $-$. Then

$$s(\gamma^{RL}_{AB}) = (a_1, a_2, \dots, a_{n-1}, a_n - 1, 1, 2 + k_1 + k_2 + k_3, a_n, a_{n-1}, \dots, a_1),$$
$$s(\gamma^{LR}_{AB}) = (a_n, a_{n-1}, \dots, a_1, 2 + k_1 + k_2 + k_3, 1, a_1 - 1, a_2, a_3, \dots, a_n).$$

Thus the relevant continued-fraction matrix products are

$$S_1 := F_{a_1} F_{a_2} \cdots F_{a_{n-1}} F_{a_n - 1} F_1 F_{2+k_1+k_2+k_3} F_{a_n} F_{a_{n-1}} \cdots F_{a_1},$$
$$S_2 := F_{a_n} F_{a_{n-1}} \cdots F_{a_1} F_{2+k_1+k_2+k_3} F_1 F_{a_1 - 1} F_{a_2} F_{a_3} \cdots F_{a_n}.$$

By Corollary 6.1.8 and Proposition 6.3.3, their $(1,1)$ entries give $|\gamma^{RL}_{AB}|$ and $|\gamma^{LR}_{AB}|$, respectively.

Put

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} := F_{a_2} F_{a_3} \cdots F_{a_{n-1}}.$$

Taking transposes gives

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} = F_{a_{n-1}} F_{a_{n-2}} \cdots F_{a_2}.$$

Therefore the two products can be rewritten as

$$S_1 = F_{a_1} \begin{bmatrix} p & q \\ r & s \end{bmatrix} F_{a_n - 1} F_1 F_{2+k_1+k_2+k_3} F_{a_n} \begin{bmatrix} p & r \\ q & s \end{bmatrix} F_{a_1},$$
$$S_2 = F_{a_n} \begin{bmatrix} p & r \\ q & s \end{bmatrix} F_{a_1} F_{2+k_1+k_2+k_3} F_1 F_{a_1 - 1} \begin{bmatrix} p & q \\ r & s \end{bmatrix} F_{a_n}.$$

A direct multiplication gives the same $(1,1)$ entry for both matrices, namely

$$(3 + k_1 + k_2 + k_3)\big(a_1(a_n p + q) + (a_n r + s)\big)^2.$$

Hence $|\gamma^{RL}_{AB}| = |\gamma^{LR}_{AB}|$ in this case.

If the second sign attached to γ^R_{AM} is $+$, the two displayed sign sequences are reversed. Repeating the computation above with the roles of S_1 and S_2 interchanged gives the same common $(1,1)$ entry, and the equality follows. □

LEMMA 6.3.15. *For lattice points A and B,*

$$|\gamma^{RL}_{AB}| = |\gamma^{LR}_{AB}| \geq |\gamma^L_{AB}| = |\gamma^R_{AB}|.$$

PROOF. The equalities $|\gamma^{RL}_{AB}| = |\gamma^{LR}_{AB}|$ and $|\gamma^L_{AB}| = |\gamma^R_{AB}|$ follow from Lemmas 6.3.8, 6.3.13, and 6.3.14. It remains to show

$$|\gamma^{LR}_{AB}| \geq |\gamma^L_{AB}|.$$

If the open segment AB contains no marked point or exactly one marked point, then the same homotopy argument as in Lemma 6.3.13 gives equality. Hence assume that the open segment AB contains at least two marked points.

Apply Theorem 6.3.11 to the crossing of

$$\gamma_1 = \gamma_{AB}^{RL}, \qquad \gamma_2 = \gamma_{AB}^{LR}.$$

Let the four resolved arcs be $\gamma_3, \gamma_4, \gamma_5, \gamma_6$. Then $\gamma_3 = \gamma_{AB}^{R}$ and $\gamma_4 = \gamma_{AB}^{L}$, and therefore

$$|\gamma_{AB}^{RL}|\,|\gamma_{AB}^{LR}| = |\gamma_{AB}^{R}|\,|\gamma_{AB}^{L}| + |\gamma_5|\,|\gamma_6|.$$

See Figure 14. Using the equalities already proved, this becomes

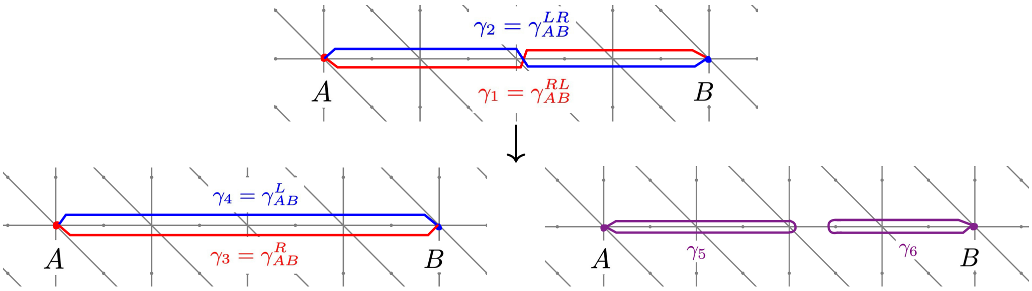


FIGURE 14. Skein resolution of γ_{AB}^{RL} and γ_{AB}^{LR}

$$|\gamma_{AB}^{LR}|^2 = |\gamma_{AB}^{L}|^2 + |\gamma_5|\,|\gamma_6|.$$

The product $|\gamma_5|\,|\gamma_6|$ can vanish only if one of the resolved curves is a contractible loop based at a common endpoint. Since $|\gamma_5| \neq 0$, this would force γ_6 to be such a contractible loop. But then the open segment AB would contain no marked point other than the midpoint, contradicting the assumption that it contains at least two marked points. Hence in the present case the inequality is in fact strict. The asserted weak inequality follows. $\square$

PROOF OF THEOREM 6.3.7. Let γ be a generalized arc from A to B realizing the GM distance. By Lemma 6.3.10, we may assume that γ has no non-contractible self-intersection. We now tighten γ within its homotopy class in the complement of the marked points of $\widetilde{\mathbb{R}^2}$, so that it almost touches the marked points; see Figure 15. After this tightening, the path of γ consists

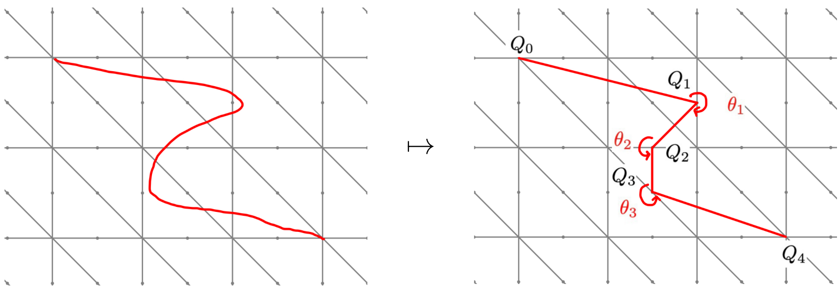


FIGURE 15. Tightening a curve

of $r \geq 1$ straight segments connecting marked points of $\widetilde{\mathbb{R}^2}$. Write the marked points through which it passes as

$$A = Q_0, Q_1, \ldots, Q_r = B.$$

Between the r line segments there are $r-1$ turning angles, which we denote by $\theta_1, \ldots, \theta_{r-1}$. Each θ_i lies in

$$[-2\pi, -\pi] \cup [\pi, 2\pi],$$

for otherwise γ could be tightened further.

If $r = 1$, then the tightened arc is a single straight segment from A to B. Hence the original generalized arc is homotopic to one of the two perturbations γ_{AB}^{L} or γ_{AB}^{R}, and the conclusion follows. Hence assume $r \geq 2$. Observe that γ is homotopic to γ_{AB}^{L} or γ_{AB}^{R} if and only if all turning angles are $-\pi$, or all turning angles are π. Interchanging A and B if necessary, we choose the convention so that all turning angles of γ_{AB}^{L} are $-\pi$.

First suppose that $\theta_1 > 0$. Assume inductively that

$$\theta_1 = \cdots = \theta_i = \pi.$$

We show that θ_{i+1} must also be π. There are three alternatives to exclude:

(1) $\theta_{i+1} \in [-2\pi, -\pi]$;
(2) $\theta_{i+1} \in (\pi, 2\pi)$;
(3) $\theta_{i+1} = 2\pi$.

In each case we construct another arc δ crossing γ, apply the skein relation, and obtain an arc γ' from A to B with $|\gamma'| < |\gamma|$. This contradicts the choice of γ as a distance-realizing arc.

We begin with case (1). Let AQ be the line segment for which A is one endpoint and Q_{i+1} is the midpoint; then the other endpoint Q is again a lattice point. Consider the arc γ^L_{AQ}. The arcs γ and γ^L_{AQ} necessarily cross near Q_{i+1}, as in Figure 16. Resolve this crossing skein-theoretically. Applying Theorem 6.3.11 with $\gamma_1 = \gamma$ and $\gamma_2 = \gamma^L_{AQ}$ at the crossing near Q_{i+1} gives $\gamma_3 = \gamma^{RL}_{QA}$.

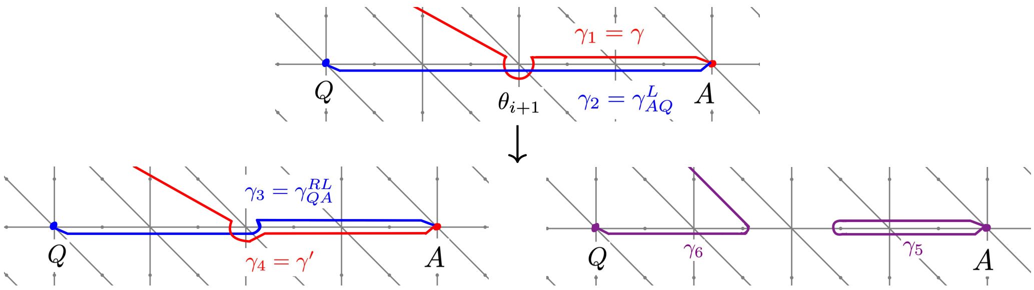


FIGURE 16. Skein resolution in case (1)

Write $\gamma_4 = \gamma'$. Then γ' is a generalized arc with endpoints A and B, and

$$|\gamma|\,|\gamma^L_{AQ}| = |\gamma^{RL}_{QA}|\,|\gamma'| + |\gamma_5|\,|\gamma_6|.$$

By Lemma 6.3.15, $|\gamma^L_{AQ}| < |\gamma^{RL}_{QA}|$. Hence

$$|\gamma'| = \frac{|\gamma|\,|\gamma^L_{AQ}| - |\gamma_5|\,|\gamma_6|}{|\gamma^{RL}_{QA}|} \leq \frac{|\gamma|\,|\gamma^L_{AQ}|}{|\gamma^{RL}_{QA}|} < |\gamma|,$$

contradicting the minimality of γ.

Next consider case (2). Again use the arc γ^L_{AQ}. The arcs γ and γ^L_{AQ} necessarily cross near Q_{i+1}, as in Figure 17. Resolving this crossing gives $\gamma_3 = \gamma^{RL}_{AQ}$ and an arc $\gamma_4 = \gamma'$ from A to B. The skein relation gives

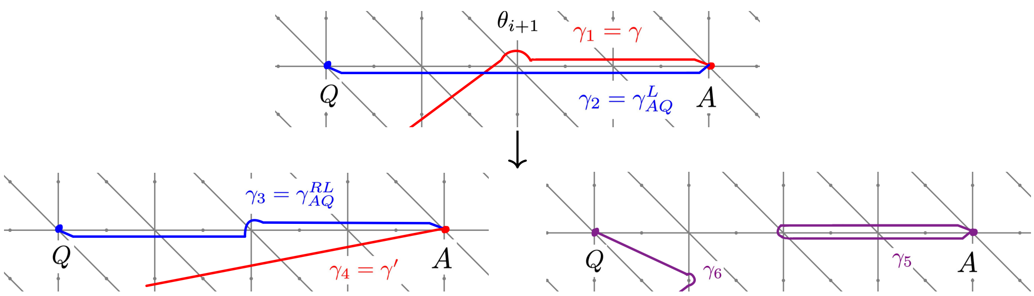


FIGURE 17. Skein resolution in case (2)

$$|\gamma|\,|\gamma^L_{AQ}| = |\gamma^{RL}_{AQ}|\,|\gamma'| + |\gamma_5|\,|\gamma_6|.$$

Since $|\gamma^L_{AQ}| < |\gamma^{RL}_{AQ}|$ by Lemma 6.3.15, the same estimate as above yields $|\gamma'| < |\gamma|$, again a contradiction.

It remains to exclude case (3). If $\theta_{i+2} = \theta_{i+3} = \cdots = \theta_{2i} = \pi$, then the curve starting at A turns back at Q_{i+1} and passes near the endpoint A. In that situation, connecting directly from the endpoint A to the returning part of γ gives a generalized arc γ' whose fence poset is a proper subposet of P_γ, and hence $|\gamma'| < |\gamma|$. Thus we may assume that there exists $2 \le j \le i$ with $\theta_{i+j} \neq \pi$.

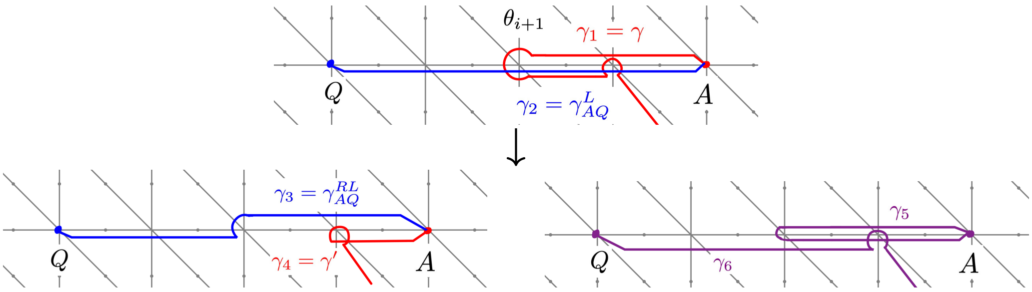


FIGURE 18. Skein resolution in case (3)

Move γ^L_{AQ} by homotopy into minimal intersection position with γ so that the two arcs intersect near the point $Q_{i-j} = Q_{i+j}$. Applying Theorem 6.3.11 with $\gamma_1 = \gamma$ and $\gamma_2 = \gamma^L_{AQ}$ gives $\gamma_5 = \gamma^{RL}_{AQ}$. If $\gamma_6 = \gamma'$, then the same skein estimate gives

$$|\gamma'| \le \frac{|\gamma|\,|\gamma^L_{AQ}|}{|\gamma^{RL}_{AQ}|} < |\gamma|,$$

contradicting minimality. Thus case (3) cannot occur.

We have shown that if $\theta_1 > 0$, then every subsequent angle must be π, and consequently $\gamma = \gamma^R_{AB}$. If $\theta_1 < 0$, the same argument applies after replacing π by $-\pi$ and interchanging L and R. Therefore every distance-realizing arc is homotopic to either γ^R_{AB} or γ^L_{AB}. Lemma 6.3.8 gives

$$d(A,B) = |\gamma^R_{AB}| = |\gamma^L_{AB}|,$$

as required. □

REMARK 6.3.16. All skein relations used in the proof of Theorem 6.3.7 come from type 0 resolutions.

In fact, when Q_{i+1} is a lattice point, one can also obtain a shorter arc γ' by resolving against a different auxiliary arc: in cases (1) and (3), take $\delta = \gamma^L_{AQ_{i+1}}$, while in case (2), take $\delta = \gamma^L_{AE}$, where E is the lattice point adjacent to Q_{i+1} on the extension of the segment AQ_{i+1}. In that alternative argument, cases (1) and (3) use skein relations coming from type 1 resolutions, and case (2) uses one coming from a type 0 resolution. In any case, no type 2 resolution is needed for the proof within the scope of this text.

We have defined GM length and GM distance purely in terms of curves. The following theorem explains how this construction recovers generalized Markov numbers. Its proof will be given later, after the generalized Cohn matrices have been introduced.

THEOREM 6.3.17. *Fix* $(k_1,k_2,k_3) \in \mathbb{Z}^3_{\ge 0}$ *and* $\sigma \in \mathfrak{S}_3$. *Let* p *and* q *be coprime nonnegative integers, and put* $A = (0,0)$ *and* $B = (q,p)$. *Then*

$$|\gamma^R_{AB}| = |\gamma^L_{AB}| = m_{p/q}.$$

In particular, $d(A,B) = m_{p/q}$.

CHAPTER 7

Generalized Cohn Matrices

In Chapter 6 we described generalized Markov numbers in terms of combinatorial and geometric objects such as fence posets, skein relations, GM lengths, and GM distances of curves. In this chapter we translate this information into the language of 2×2 matrices and reinterpret the theory of generalized Markov numbers from a matrix-theoretic viewpoint. The central objects are generalized Cohn matrices. They generalize the Cohn matrices that play an important role in the classical theory, and they collect the numerical and curve-theoretic information introduced in the preceding chapters within a single framework.

We first define the generalized Cohn tree and construct, in a systematic way, the generalized Cohn matrix attached to each reduced fraction. We then show that its entries are described explicitly by generalized Markov numbers and characteristic numbers. This makes clear that the GC matrix encodes the arithmetic data introduced above. We also prove a relation among characteristic numbers, and finally introduce generalized strongly admissible sequences in order to express GC matrices as products of elementary continued-fraction matrices. In this way the arithmetic information prepared in Chapter 5 and the combinatorial and geometric information obtained in Chapter 6 are unified through matrix representations.

The important point is that the relations among generalized Markov numbers, characteristic numbers, curves, and continued-fraction data are reflected simultaneously in a single object, the GC matrix. For this reason, GC matrices are useful tools for analyzing these relations. In fact, two important results postponed in Chapters 5 and 6 will be proved by computations with GC matrices. The matrix representation obtained here also provides the basis for defining the generalized discrete Markov spectrum and for realizing its values explicitly in the next chapter.

The material in this chapter is based on [**GM23b, GMS24, Gyo25**]. The definition of generalized Cohn matrices in [**GM23b, GMS24**] is slightly different from the continued-fraction matrix notation used in this text. We therefore adjust the definition so that it is compatible with the notation used below. This is only a notational adjustment, not a change in the substance of the earlier work.

1. Definitions and Examples

Fix $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$. Define $C_{\frac{0}{1}}, C_{\frac{1}{1}}, C_{\frac{1}{0}}$ by

$$C_{\frac{0}{1}} = \begin{bmatrix} 3+k_1+k_2+k_3 & -(3+k_1+k_2+k_3)k_{\sigma(1)}-1 \\ 1 & -k_{\sigma(1)} \end{bmatrix},$$
$$C_{\frac{1}{1}} = \begin{bmatrix} (3+k_1+k_2+k_3)(k_{\sigma(2)}+2)-k_{\sigma(2)}-1 & 2+k_1+k_2+k_3 \\ k_{\sigma(2)}+2 & 1 \end{bmatrix},$$
$$C_{\frac{1}{0}} = \begin{bmatrix} 2+k_1+k_2+k_3-k_{\sigma(3)} & 1+k_1+k_2+k_3-k_{\sigma(3)} \\ 1 & 1 \end{bmatrix}.$$

For every $(r,t,s) \in \mathrm{F}\mathbb{T}$, define recursively

$$C_{r\oplus t} := C_r C_t - D_s, \quad C_{t\oplus s} := C_t C_s - D_r$$

where

$$D_r = \begin{bmatrix} k_{i_r} & k_{i_r}(3+k_1+k_2+k_3) \\ 0 & k_{i_r} \end{bmatrix}$$

The matrix C_t is called the (k_1, k_2, k_3, σ)-*generalized Cohn matrix*, or simply the *GC matrix*. We also define the (k_1, k_2, k_3, σ)-*generalized Cohn tree*, or *GC tree*, by

$$\mathrm{Co}\mathbb{T}(k_1, k_2, k_3, \sigma) := \mathrm{F}\mathbb{T}|_{(r,t,s)\mapsto(C_r, C_t, C_s)}.$$

Each vertex of this tree is called a (k_1, k_2, k_3, σ)-*generalized Cohn triple*, or simply a *GC triple*.

Example 7.1.1. The first few vertices of $\mathrm{Co}\mathbb{T}(1, 2, 0, \mathrm{id})$ are as follows:

$$\left(\begin{bmatrix}6 & -7\\1 & -1\end{bmatrix}, \begin{bmatrix}21 & 5\\4 & 1\end{bmatrix}, \begin{bmatrix}5 & 4\\1 & 1\end{bmatrix}\right)$$

- $\left(\begin{bmatrix}21 & 5\\4 & 1\end{bmatrix}, \begin{bmatrix}109 & 83\\21 & 16\end{bmatrix}, \begin{bmatrix}5 & 4\\1 & 1\end{bmatrix}\right)$
  - $\left(\begin{bmatrix}109 & 83\\21 & 16\end{bmatrix}, \begin{bmatrix}626 & 507\\121 & 98\end{bmatrix}, \begin{bmatrix}5 & 4\\1 & 1\end{bmatrix}\right)$
  - $\left(\begin{bmatrix}21 & 5\\4 & 1\end{bmatrix}, \begin{bmatrix}2394 & 1823\\457 & 348\end{bmatrix}, \begin{bmatrix}109 & 83\\21 & 16\end{bmatrix}\right)$
- $\left(\begin{bmatrix}6 & -7\\1 & -1\end{bmatrix}, \begin{bmatrix}98 & 23\\17 & 4\end{bmatrix}, \begin{bmatrix}21 & 5\\4 & 1\end{bmatrix}\right)$
  - $\left(\begin{bmatrix}98 & 23\\17 & 4\end{bmatrix}, \begin{bmatrix}2149 & 507\\373 & 88\end{bmatrix}, \begin{bmatrix}21 & 5\\4 & 1\end{bmatrix}\right)$
  - $\left(\begin{bmatrix}6 & -7\\1 & -1\end{bmatrix}, \begin{bmatrix}367 & 98\\81 & 17\end{bmatrix}, \begin{bmatrix}98 & 23\\17 & 4\end{bmatrix}\right)$

2. Description of the Entries in Terms of Generalized Markov Numbers and Characteristic Numbers

In what follows we abbreviate k_{i_t} to k_t. The goal of this section is to prove the following explicit description of the entries of a GC matrix.

Theorem 7.2.1. *For every reduced fraction $t \in [0, \infty]$,*

$$C_t = \begin{bmatrix}(3 + k_1 + k_2 + k_3)m_t - k_t - u_t & \dfrac{(3 + k_1 + k_2 + k_3)m_t u_t - k_t u_t - u_t^2 - 1}{m_t}\\ m_t & u_t\end{bmatrix}.$$

For notational brevity, put $K := 3 + k_1 + k_2 + k_3$. We begin with the trace and determinant.

Proposition 7.2.2. *For every $t \in [0, \infty]$, the following hold.*

(1) $\mathrm{tr}(C_t) = K(C_t)_{21} - k_t$,
(2) $C_t \in SL(2, \mathbb{Z})$.

We first record two lemmas for the proof of Proposition 7.2.2.

Lemma 7.2.3. *Let $A, B \in SL(2, \mathbb{Z})$. Then*

(1) $\mathrm{tr}(A) = \mathrm{tr}(A^{-1})$
(2) $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})$
(3) $A^2 = \mathrm{tr}(A)A - E_2$, *where E_2 is the 2×2 identity matrix. This is the special form of the Cayley–Hamilton theorem for matrices in $SL(2, \mathbb{Z})$.*

Proof. All three identities follow by a direct entry-wise computation. □

Lemma 7.2.4. *Let $M = \left[\begin{smallmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{smallmatrix}\right] \in SL(2, \mathbb{Z})$ satisfy $\mathrm{tr}(M) = Km_{21} - k_t$. Then*

$$M\begin{bmatrix}0 & K\\0 & 0\end{bmatrix}M = (\mathrm{tr}(M) + k_t)M + \begin{bmatrix}0 & K\\0 & 0\end{bmatrix},$$

$$M^{-1}\begin{bmatrix}0 & K\\0 & 0\end{bmatrix}M^{-1} = -(\mathrm{tr}(M^{-1}) + k_t)M^{-1} + \begin{bmatrix}0 & K\\0 & 0\end{bmatrix}.$$

Proof. We prove the first identity. Since

$$\begin{bmatrix}0 & K\\0 & 0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & K\end{bmatrix},$$

we have

$$M \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} M = M \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & K \end{bmatrix} M = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} \begin{bmatrix} Km_{21} & Km_{22} \end{bmatrix} = K \begin{bmatrix} m_{11}m_{21} & m_{11}m_{22} \\ m_{21}^2 & m_{21}m_{22} \end{bmatrix}.$$

On the other hand,

$$(\mathrm{tr}(M) + k_t)M + \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} = Km_{21}M + \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} = K \begin{bmatrix} m_{11}m_{21} & m_{12}m_{21} + 1 \\ m_{21}^2 & m_{21}m_{22} \end{bmatrix}.$$

These two matrices are equal because $m_{11}m_{22} - m_{12}m_{21} = 1$.

For the second identity, write $M^{-1} = \left[\begin{smallmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{smallmatrix}\right]$. Then

$$M^{-1} \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} M^{-1} = \begin{bmatrix} m_{22} \\ -m_{21} \end{bmatrix} \begin{bmatrix} -Km_{21} & Km_{11} \end{bmatrix} = K \begin{bmatrix} -m_{21}m_{22} & m_{11}m_{22} \\ m_{21}^2 & -m_{11}m_{21} \end{bmatrix}.$$

Using $\mathrm{tr}(M^{-1}) = \mathrm{tr}(M) = Km_{21} - k_t$, the right-hand side of the desired identity is

$$-(\mathrm{tr}(M^{-1}) + k_t)M^{-1} + \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} = -Km_{21}M^{-1} + \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} = K \begin{bmatrix} -m_{21}m_{22} & m_{12}m_{21} + 1 \\ m_{21}^2 & -m_{11}m_{21} \end{bmatrix}.$$

Again this agrees with the preceding matrix because $m_{11}m_{22} - m_{12}m_{21} = 1$. This proves the lemma. □

Proof of Proposition 7.2.2. We first prove (1). For $C_{\frac{0}{1}}, C_{\frac{1}{1}}, C_{\frac{1}{0}}$ the assertion follows by direct computation. The remaining cases are proved by induction on the distance from the initial vertex of the GC tree. Assume that a GC triple (C_r, C_t, C_s) satisfies (1). It suffices to prove

$$\mathrm{tr}(C_{r\oplus t}) = K(C_{r\oplus t})_{21} - k_{r\oplus t}, \quad \mathrm{tr}(C_{t\oplus s}) = K(C_{t\oplus s})_{21} - k_{t\oplus s}.$$

Notice that $k_{r\oplus t} = k_s$ and $k_{t\oplus s} = k_r$. We prove only the first identity; the other is analogous. Since $C_t = C_rC_s - D_t$,

$$\begin{aligned}
\mathrm{tr}(C_{r\oplus t}) &= \mathrm{tr}(C_rC_t - D_s) \\
&= \mathrm{tr}(C_r(C_rC_s - D_t)) - 2k_s \\
&= \mathrm{tr}(C_r^2C_s) - \mathrm{tr}(C_rD_t) - 2k_s \\
&\overset{\text{Lemma 7.2.3 (2)}}{=} \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}(C_rC_s^{-1}C_r^{-1}) - \mathrm{tr}(C_rD_t) - 2k_s \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}(C_s) - \mathrm{tr}(C_rD_t) - 2k_s.
\end{aligned}$$

We now rewrite the right-hand side of this expression. Put $C_r = \left[\begin{smallmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{smallmatrix}\right]$. Then

$$\mathrm{tr}(D_tC_r^{-1}) = \mathrm{tr}\left(\begin{bmatrix} r_{22}k_t - r_{21}Kk_t & * \\ * & r_{11}Kk_t \end{bmatrix}\right) = k_t\mathrm{tr}(C_r) - Kk_tr_{21} = -k_rk_t$$

where we used the induction hypothesis for C_r. Hence

$$\begin{aligned}
&\mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}(C_rD_t) - \mathrm{tr}(C_s) - 2k_s \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}(C_rD_t) - \mathrm{tr}(C_s) - 2k_s + k_rk_t - k_rk_t \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}(C_rD_t) - \mathrm{tr}(C_r^{-1}D_t) - \mathrm{tr}(C_s) - k_rk_t - 2k_s \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}((C_r + C_r^{-1})D_t) - \mathrm{tr}(C_s) - k_rk_t - 2k_s \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s) - \mathrm{tr}(C_r)\mathrm{tr}(D_t) - \mathrm{tr}(C_s) - k_rk_t - 2k_s \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_rC_s - D_t) - \mathrm{tr}(C_s) - k_rk_t - 2k_s \\
&= \mathrm{tr}(C_r)\mathrm{tr}(C_t) - \mathrm{tr}(C_s) - k_rk_t - 2k_s \\
&= \left(\begin{bmatrix} 0 & K \end{bmatrix} C_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} - k_r\right)\left(\begin{bmatrix} 0 & K \end{bmatrix} C_t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - k_t\right) \\
&\quad - \left(\begin{bmatrix} 0 & K \end{bmatrix} C_s \begin{bmatrix} 1 \\ 0 \end{bmatrix} - k_s\right) - k_rk_t - 2k_s \\
&= \begin{bmatrix} 0 & K \end{bmatrix} C_r \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} C_rC_s \begin{bmatrix} 1 \\ 0 \end{bmatrix} - k_r \begin{bmatrix} 0 & K \end{bmatrix} C_rC_s \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$- k_t \begin{bmatrix}0 & K\end{bmatrix} C_r \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0 & K\end{bmatrix} C_s \begin{bmatrix}1\\0\end{bmatrix} - k_s$$

$$\overset{\text{Lemma 7.2.4}}{=} \begin{bmatrix}0 & K\end{bmatrix} \left((\mathrm{tr} C_r + k_r) C_r + \begin{bmatrix}0 & K\\0 & 0\end{bmatrix} \right) C_s \begin{bmatrix}1\\0\end{bmatrix} - k_r \begin{bmatrix}0 & K\end{bmatrix} C_r C_s \begin{bmatrix}1\\0\end{bmatrix}$$

$$- k_t \begin{bmatrix}0 & K\end{bmatrix} C_r \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0 & K\end{bmatrix} C_s \begin{bmatrix}1\\0\end{bmatrix} - k_s$$

$$\overset{\text{Lemma 7.2.3 (3)}}{=} \begin{bmatrix}0 & K\end{bmatrix} C_r^2 C_s \begin{bmatrix}1\\0\end{bmatrix} - k_t \begin{bmatrix}0 & K\end{bmatrix} C_r \begin{bmatrix}1\\0\end{bmatrix} - k_s$$

$$= \begin{bmatrix}0 & K\end{bmatrix} (C_r^2 C_s - C_r D_t - D_s) \begin{bmatrix}1\\0\end{bmatrix} - k_s$$

$$= \begin{bmatrix}0 & K\end{bmatrix} (C_r C_t - D_s) \begin{bmatrix}1\\0\end{bmatrix} - k_s = K(C_{r\oplus t})_{21} - k_s.$$

This proves the required trace identity. We next prove (2). It suffices to show $\det(C_{r\oplus t}) = \det(C_{t\oplus s}) = 1$, and again we prove only the first equality. Write $C_r C_t = \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{smallmatrix}\right]$. Then

$$\begin{aligned}\det(C_{r\oplus t}) &= \det(C_r C_t - D_s)\\ &= (x_{11} - k_s)(x_{22} - k_s) - x_{21}(x_{12} - K k_s)\\ &= \det(C_r C_t) - k_s \mathrm{tr}(C_r C_t) + k_s^2 + K k_s x_{21} = \det(C_r C_t) = 1.\end{aligned}$$

This completes the proof. □

We now use this to prove the following proposition.

Proposition 7.2.5. *For every $t \in [0, \infty]$, one has $(C_t)_{21} = m_t$.*

Proof. For $t = \frac{0}{1}, \frac{1}{1}, \frac{1}{0}$ this is immediate by direct computation. We prove the remaining cases by induction on the distance from the initial vertex of the GC tree. Assume that the assertion holds for a GC triple (C_r, C_t, C_s). We show that it holds for the left child; the proof for the right child is analogous.

By the induction hypothesis and Proposition 7.2.2 (1), we may write

$$C_r = \begin{bmatrix} K m_r - k_r - (C_r)_{22} & * \\ m_r & (C_r)_{22}\end{bmatrix}, \qquad C_t = \begin{bmatrix} K m_t - k_t - (C_t)_{22} & * \\ m_t & (C_t)_{22}\end{bmatrix}.$$

Since $C_{r\oplus t} = C_r C_t - D_s$, comparison of the $(2,1)$ entries gives

$$\begin{aligned}(C_{r\oplus t})_{21} &= m_r (K m_t - k_t - (C_t)_{22}) + (C_r)_{22} m_t\\ &= K m_r m_t - m_r k_t - m_t\left(k_r + (C_r)_{22}\right) + m_t (C_r)_{22} - m_r (C_t)_{22}\\ &= K m_r m_t - m_r k_t - m_t k_r - \left(m_r (C_t)_{22} - m_t(k_r + (C_r)_{22})\right). \qquad (7.2.1)\end{aligned}$$

The relation $C_s = C_r^{-1}(C_t + D_t)$, together with the same matrix forms, shows that $m_s = m_r (C_t)_{22} - m_t(k_r + (C_r)_{22})$. Substituting this into (7.2.1), we obtain $(C_{r\oplus t})_{21} = K m_r m_t - m_r k_t - m_t k_r - m_s$. Because (m_r, m_t, m_s) satisfies the GM equation

$$m_r^2 + m_t^2 + m_s^2 + k_r m_t m_s + k_t m_s m_r + k_s m_r m_t = K m_r m_t m_s,$$

we may rewrite the last expression as $(C_{r\oplus t})_{21} = \frac{m_r^2 + k_s m_r m_t + m_t^2}{m_s} = m_{r\oplus t}$. This completes the induction step. □

It remains to identify the $(2,2)$ entry of C_t.

Proposition 7.2.6. *For every $t \in [0, \infty]$, one has $(C_t)_{22} = u_t$.*

We introduce the following index.

Definition 7.2.7. For $t \in [0, \infty] \cap \mathbb{Q}$, define $I_t := \frac{(C_t)_{22}}{(C_t)_{21}}$. This number is called the *index* of C_t.

Proposition 7.2.8. *The index is strictly increasing: if $s < t$, then $I_s < I_t$. Here $\frac{1}{0}$ is regarded as larger than every rational number.*

PROOF. It suffices to prove $I_r < I_t < I_s$ for each Farey triple $(r,t,s) \in \mathbb{FT}$. Write the entries of C_r, C_t, C_s as r_{ij}, t_{ij}, s_{ij}, respectively. We first prove $I_t < I_s$. Since $C_t = C_r C_s - D_t$, we have $C_r = (C_t + D_t)C_s^{-1}$. Comparing the $(2,1)$ entries gives

$$r_{21} = s_{22}t_{21} - (t_{22} + k_t)s_{21} \leq s_{22}t_{21} - t_{22}s_{21}.$$

Therefore

$$0 < \frac{r_{21}}{t_{21}s_{21}} \leq \frac{s_{22}}{s_{21}} - \frac{t_{22}}{t_{21}} = I_s - I_t.$$

Thus $I_t < I_s$. We next prove $I_r < I_t$. From $C_t = C_r C_s - D_t$ we obtain $C_s = C_r^{-1}(C_t + D_t)$. Comparing the $(2,1)$ entries gives

$$\begin{aligned} s_{21} &= r_{11}t_{21} - r_{21}(t_{11} + k_t) \\ &= (Kr_{21} - k_r - r_{22})t_{21} - r_{21}(Kt_{21} - t_{22}) \\ &= -k_r t_{21} + t_{22}r_{21} - t_{21}r_{22} \leq t_{22}r_{21} - t_{21}r_{22}. \end{aligned}$$

Therefore

$$0 < \frac{s_{21}}{r_{21}t_{21}} \leq \frac{t_{22}}{t_{21}} - \frac{r_{22}}{r_{21}} = I_t - I_r.$$

Thus $I_r < I_t$. □

LEMMA 7.2.9. *For every reduced fraction $t \in (0, \infty]$, one has $(C_t)_{22} > 0$.*

PROOF. By the monotonicity of the index, Proposition 7.2.8, it suffices to consider $t = \frac{1}{n}$. The $(2,2)$ entry of $C_{\frac{1}{1}}$ is 1, and that of $C_{\frac{1}{2}}$ is $k_{\frac{1}{0}} + 2$, hence is larger. Assume that the $(2,2)$ entry of $C_{\frac{1}{i}}$ is positive and that the $(2,2)$ entry of $C_{\frac{1}{i+1}}$ is larger than that of $C_{\frac{1}{i}}$. We show that the $(2,2)$ entry of $C_{\frac{1}{i+2}}$ is larger than that of $C_{\frac{1}{i+1}}$, and in particular is positive. Write

$$C_{\frac{1}{i}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad C_{\frac{1}{i+1}} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \quad C_{\frac{1}{i+2}} = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}.$$

Then

$$\begin{aligned} C_{\frac{1}{i+1}} &= C_{\frac{0}{1}} C_{\frac{1}{i}} - D_{\frac{1}{i-1}} \\ &= \begin{bmatrix} K & -Kk_{\frac{0}{1}} - 1 \\ 1 & -k_{\frac{0}{1}} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} k_{\frac{1}{i-1}} & Kk_{\frac{1}{i-1}} \\ 0 & k_{\frac{1}{i-1}} \end{bmatrix} \\ &= \begin{bmatrix} * & Kb - Kk_{\frac{0}{1}}d - d - Kk_{\frac{1}{i-1}} \\ * & b - k_{\frac{0}{1}}d - k_{\frac{1}{i-1}} \end{bmatrix}. \end{aligned}$$

Therefore $b' = Kb - Kk_{\frac{0}{1}}d - d - Kk_{\frac{1}{i-1}}$ and $d' = b - k_{\frac{0}{1}}d - k_{\frac{1}{i-1}}$. By the induction hypothesis, $b - k_{\frac{0}{1}}d - k_{\frac{1}{i-1}} \geq d > 0$. Moreover, since $C_{\frac{1}{i+2}} = C_{\frac{0}{1}} C_{\frac{1}{i+1}} - D_{\frac{1}{i}}$, we have

$$\begin{aligned} d'' - d' &= b' - (k_{\frac{0}{1}} + 1)d' - k_{\frac{1}{i}} \\ &= (K - k_{\frac{0}{1}} - 1)b + (k_{\frac{0}{1}}^2 + (1-K)k_{\frac{0}{1}} - 1)d + (k_{\frac{0}{1}} + 1 - K)k_{\frac{1}{i-1}} - k_{\frac{1}{i}} \\ &\geq (K - k_{\frac{0}{1}} - 1)((k_{\frac{0}{1}} + 1)d + k_{\frac{1}{i-1}}) + (k_{\frac{0}{1}}^2 + (1-K)k_{\frac{0}{1}} - 1)d + (k_{\frac{0}{1}} + 1 - K)k_{\frac{1}{i-1}} - k_{\frac{1}{i}} \\ &= (K - 2 - k_{\frac{0}{1}})d - k_{\frac{1}{i}} \geq k_{\frac{1}{i+1}} + 1 > 0. \end{aligned}$$

This proves the claim. □

PROOF OF PROPOSITION 7.2.6. Let (r,t,s) be a Farey triple. From $C_s = C_r^{-1}(C_t + D_t)$, and from Propositions 7.2.2 (1) and 7.2.5, we may write

$$C_r = \begin{bmatrix} Km_r - k_r - (C_r)_{22} & * \\ m_r & (C_r)_{22} \end{bmatrix}, \qquad C_t = \begin{bmatrix} Km_t - k_t - (C_t)_{22} & * \\ m_t & (C_t)_{22} \end{bmatrix}.$$

Comparing the $(2,1)$ entries gives $m_s = m_r(C_t)_{22} - m_t(k_r + (C_r)_{22})$, and therefore $m_r(C_t)_{22} \equiv m_s \pmod{m_t}$. Moreover, Proposition 7.2.8 and Lemma 7.2.9 imply $0 < I_t < I_{\frac{1}{0}} = 1$. Since

$(C_t)_{21} = m_t$, this gives $0 < (C_t)_{22} < m_t$. The defining congruence and uniqueness of the characteristic number u_t now imply $u_t = (C_t)_{22}$. □

We are now ready to prove Theorem 7.2.1.

Proof of Theorem 7.2.1. This is exactly the combination of the three preceding propositions. □

3. A Relation among Characteristic Numbers

In this section we use GC matrices to prove the proposition postponed in Section 4.

Proposition 5.4.4 (recalled).
, For a reduced fraction $t \in [0,1] \cap \mathbb{Q}$, let u_t be the characteristic number with fraction label t in $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$ and put $k_t := k_{i_t}$. Let $u^*_{1/t}$ be the characteristic number with fraction label $1/t$ in $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma^*)$. Then $u^*_{\frac{1}{t}} = m_t - u_t - k_t$.

Lemma 7.3.1. *Let m_t be a (k_1,k_2,k_3)-GM number in $\mathrm{M}\mathbb{T}(k_1,k_2,k_3,\sigma)$ whose fraction label is $t \in (0,1] \cap \mathbb{Q}$. Then the following hold.*

(1) *If $i_t = \sigma(1)$, then $m_t \geq m_{\frac{2}{3}}$.*
(2) *If $i_t = \sigma(2)$, then $m_t \geq m_{\frac{1}{1}}$.*
(3) *If $i_t = \sigma(3)$, then $m_t \geq m_{\frac{1}{2}}$.*

Moreover, these bounds are sharp.

Proof. By Remark 5.1.8 and Proposition 5.2.6, whenever one passes from a vertex of the GM tree to a child vertex, the newly created GM number is larger than each GM number already appearing in the parent vertex. This immediately gives (2) and (3), and the bounds are attained at the labels $1/1$ and $1/2$, respectively.

It remains to prove (1). By the same monotonicity, it is enough to compare the numbers $m_{2/(2i+1)}$ for $i \geq 1$. The Farey triple whose second component is $2/(2i+1)$ is $\left(\frac{1}{i+1}, \frac{2}{2i+1}, \frac{1}{i}\right)$, and its parent is $\left(\frac{0}{1}, \frac{1}{i+1}, \frac{1}{i}\right)$. Hence the mutation formula gives

$$m_{\frac{2}{2i+1}} = \frac{m^2_{\frac{1}{i+1}} + k_{\sigma(1)} m_{\frac{1}{i+1}} m_{\frac{1}{i}} + m^2_{\frac{1}{i}}}{m_{\frac{0}{1}}} = m^2_{\frac{1}{i+1}} + k_{\sigma(1)} m_{\frac{1}{i+1}} m_{\frac{1}{i}} + m^2_{\frac{1}{i}}.$$

For every $i \geq 1$, the monotonicity just recalled gives $m_{1/i} < m_{1/(i+1)}$. Since the polynomial $x^2 + k_{\sigma(1)}xy + y^2$ is strictly increasing in both positive variables, we obtain

$$m_{\frac{2}{2i+1}} < m_{\frac{2}{2i+3}} \qquad (i \geq 1).$$

Thus the minimum in the case $i_t = \sigma(1)$ is attained at $i = 1$, namely at $m_{2/3}$. This also shows that the bound in (1) is sharp. □

Proof of Proposition 5.4.4. The cases $t = \frac{0}{1}$ and $t = \frac{1}{1}$ are checked directly from the initial vertices. Hence assume $t \in (0,1) \cap \mathbb{Q}$, and let (r,t,s) be the Farey triple whose second component is t. Then $(1/s, 1/t, 1/r)$ is also a Farey triple.

By the definition of the characteristic number for the dual tree, $u^*_{1/t}$ is the unique integer satisfying

$$m^*_{\frac{1}{s}} u^*_{\frac{1}{t}} \equiv m^*_{\frac{1}{r}} \pmod{m^*_{\frac{1}{t}}}, \qquad 0 < u^*_{\frac{1}{t}} < m^*_{\frac{1}{t}}.$$

By Corollary 5.3.7, $m^*_{\frac{1}{r}} = m_r$, $m^*_{\frac{1}{t}} = m_t$, and $m^*_{\frac{1}{s}} = m_s$. It is therefore enough to prove that $x := m_t - u_t - k_t$ satisfies $m_s x \equiv m_r \pmod{m_t}$ and $0 < x < m_t$. For the Cohn triple (C_r, C_t, C_s) we have $C_r = (C_t + D_t)C_s^{-1}$. Comparing the $(2,1)$ entries gives

$$m_r = m_t u_s + m_s(-u_t - k_t) \equiv m_s(-u_t - k_t) \equiv m_s(m_t - u_t - k_t) \pmod{m_t},$$

so the required congruence holds.

It remains to prove the inequalities. The upper bound $x < m_t$ is immediate from $u_t + k_t > 0$. For the lower bound, use Proposition 7.2.8: since $t < 1$, we have $\frac{u_t}{m_t} = I_t < I_{\frac{1}{1}} = \frac{1}{k_{\sigma(2)}+2}$. If $i_t = \sigma(2)$, then Lemma 7.3.1 gives $m_t \geq m_{\frac{1}{1}} = k_{\sigma(2)} + 2$, and hence

$$m_t - u_t - k_t > (k_{\sigma(2)} + 2)\left(1 - \frac{1}{k_{\sigma(2)} + 2}\right) - k_{\sigma(2)} = 1 > 0.$$

If $i_t = \sigma(3)$, then

$$m_t \geq m_{\frac{1}{2}} = Kk_{\sigma(2)} + 2K - k_{\sigma(1)}k_{\sigma(2)} - 2k_{\sigma(1)} - k_{\sigma(2)} - 1.$$

Therefore

$$\begin{aligned} m_t - u_t - k_t &> (Kk_{\sigma(2)} + 2K - k_{\sigma(1)}k_{\sigma(2)} - 2k_{\sigma(1)} - k_{\sigma(2)} - 1)\left(1 - \frac{1}{k_{\sigma(2)} + 2}\right) - k_{\sigma(3)} \\ &\geq \frac{1}{2}(Kk_{\sigma(2)} + 2K - k_{\sigma(1)}k_{\sigma(2)} - 2k_{\sigma(1)} - k_{\sigma(2)} - 1) - k_{\sigma(3)} \\ &= \frac{1}{2}(k_{\sigma(2)}^2 + k_{\sigma(3)}k_{\sigma(2)} + 4k_{\sigma(2)} + 5) > 0. \end{aligned}$$

If $i_t = \sigma(1)$, then

$$\begin{aligned} m_t &\geq m_{\frac{2}{3}} \\ &= (Kk_{\sigma(2)} + 2K - k_{\sigma(1)}k_{\sigma(2)} - 2k_{\sigma(1)} - k_{\sigma(2)} - 1)\big(K(k_{\sigma(2)} + 2) - k_{\sigma(2)} - 1\big) \\ &\quad + (K - k_{\sigma(1)} - k_{\sigma(3)} - 1)(k_{\sigma(2)} + 2). \end{aligned}$$

Substituting $K = 3 + k_1 + k_2 + k_3$ gives

$$\begin{aligned} m_t \geq\ & k_{\sigma(1)}k_{\sigma(3)}k_{\sigma(2)}^2 + 4k_{\sigma(1)}k_{\sigma(3)}k_{\sigma(2)} + 4k_{\sigma(1)}k_{\sigma(3)} \\ & + k_{\sigma(1)}k_{\sigma(2)}^3 + 6k_{\sigma(1)}k_{\sigma(2)}^2 + 13k_{\sigma(1)}k_{\sigma(2)} + 10k_{\sigma(1)} \\ & + k_{\sigma(3)}^2k_{\sigma(2)}^2 + 4k_{\sigma(3)}^2k_{\sigma(2)} + 4k_{\sigma(3)}^2 \\ & + 2k_{\sigma(3)}k_{\sigma(2)}^3 + 12k_{\sigma(3)}k_{\sigma(2)}^2 + 26k_{\sigma(3)}k_{\sigma(2)} + 20k_{\sigma(3)} \\ & + k_{\sigma(2)}^4 + 8k_{\sigma(2)}^3 + 27k_{\sigma(2)}^2 + 44k_{\sigma(2)} + 29. \end{aligned}$$

Using the same estimate for u_t/m_t as above, we obtain

$$\begin{aligned} m_t - u_t - k_t >\ & \frac{1}{2}\big(k_{\sigma(1)}k_{\sigma(3)}k_{\sigma(2)}^2 + 4k_{\sigma(1)}k_{\sigma(3)}k_{\sigma(2)} + 4k_{\sigma(1)}k_{\sigma(3)} \\ & + k_{\sigma(1)}k_{\sigma(2)}^3 + 6k_{\sigma(1)}k_{\sigma(2)}^2 + 13k_{\sigma(1)}k_{\sigma(2)} + 10k_{\sigma(1)} \\ & + k_{\sigma(3)}^2k_{\sigma(2)}^2 + 4k_{\sigma(3)}^2k_{\sigma(2)} + 4k_{\sigma(3)}^2 + 2k_{\sigma(3)}k_{\sigma(2)}^3 \\ & + 12k_{\sigma(3)}k_{\sigma(2)}^2 + 26k_{\sigma(3)}k_{\sigma(2)} + 20k_{\sigma(3)} \\ & + k_{\sigma(2)}^4 + 8k_{\sigma(2)}^3 + 27k_{\sigma(2)}^2 + 44k_{\sigma(2)} + 29\big) - k_{\sigma(1)} \\ =\ & \frac{1}{2}\big(k_{\sigma(1)}k_{\sigma(3)}k_{\sigma(2)}^2 + 4k_{\sigma(1)}k_{\sigma(3)}k_{\sigma(2)} + 4k_{\sigma(1)}k_{\sigma(3)} \\ & + k_{\sigma(1)}k_{\sigma(2)}^3 + 6k_{\sigma(1)}k_{\sigma(2)}^2 + 13k_{\sigma(1)}k_{\sigma(2)} + 8k_{\sigma(1)} \\ & + k_{\sigma(3)}^2k_{\sigma(2)}^2 + 4k_{\sigma(3)}^2k_{\sigma(2)} + 4k_{\sigma(3)}^2 + 2k_{\sigma(3)}k_{\sigma(2)}^3 \\ & + 12k_{\sigma(3)}k_{\sigma(2)}^2 + 26k_{\sigma(3)}k_{\sigma(2)} + 20k_{\sigma(3)} \\ & + k_{\sigma(2)}^4 + 8k_{\sigma(2)}^3 + 27k_{\sigma(2)}^2 + 44k_{\sigma(2)} + 29\big) > 0. \end{aligned}$$

This proves $0 < x < m_t$, and therefore the proposition follows. ∎

4. Description of Generalized Cohn Matrices by Generalized Strongly Admissible Sequences

In this section we prove that a GC matrix can in fact be decomposed as a product of elementary matrices.

Let $t = \frac{p}{q}$ be a positive reduced fraction. In $\widetilde{\mathbb{R}^2}$ take the two points $A = (0,0)$ and $B = (q,p)$, and consider the curve segment γ_{AB}^{L} joining them. In this section we denote this curve by L_t in order to emphasize its dependence on the fraction t. Let $\overline{L_t}$ be the curve segment obtained by shifting the two endpoints slightly to the left from A and B, respectively; see Figure 1. The left endpoint is regarded as passing through the lower-left edge, whereas the right endpoint is regarded as staying in the interior of the upper-right edge. This subtle difference between the two endpoints matters when applying the sign rules.

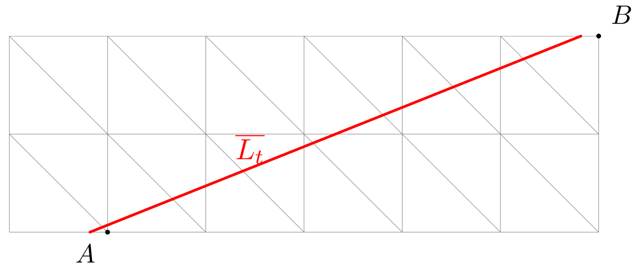


Figure 1. The segment $\overline{L_t}$ corresponding to $t = \frac{2}{5}$

Remark 7.4.1. Equivalently, one may obtain $\overline{L_t}$ using the once-punctured torus. Project L_t in $\widetilde{\mathbb{R}^2}$ to the triangulated once-punctured torus, push the portion of the arc near the puncture, together with a small neighborhood, slightly upward to obtain a loop, and then lift this loop back to $\widetilde{\mathbb{R}^2}$. Figure 2 illustrates the case $t = \frac{1}{1}$.

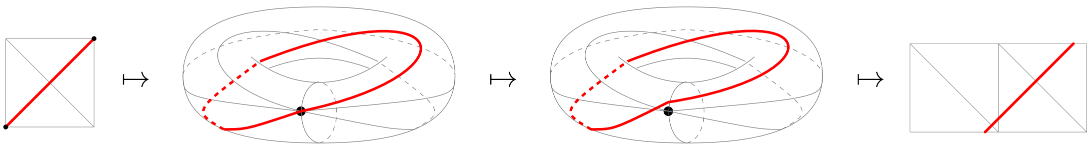


Figure 2. Interpretation of $\overline{L_t}$ as a loop on the torus for $t = \frac{1}{1}$

Now fix $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$. We define the (k_1, k_2, k_3, σ)-*generalized strongly admissible sequence* $s(t)$.[1] For the endpoints, set

$$s(\tfrac{0}{1}) = (1 + k_{\sigma(2)} + k_{\sigma(3)}, 1), \quad s(\tfrac{1}{0}) = (1 + k_{\sigma(1)} + k_{\sigma(2)}, 1).$$

For a reduced fraction $t \in (0, \infty)$, define $s(t)$ as follows.

(1) Orient $\overline{L_t}$ from the lower left to the upper right. List, in the order in which $\overline{L_t}$ crosses them, the signs assigned by the triangle-crossing rule and the edge-crossing rule.

(2) Form the integer sequence $s(t) = (a_1, \dots, a_\ell)$ from the lengths of the maximal consecutive blocks of equal signs appearing in (1).

Example 7.4.2. Let $(k_1, k_2, k_3, \sigma) = (1, 2, 0, \mathrm{id})$ and $t = \frac{2}{5}$. Figure 3 shows the signs along $\overline{L_t}$, and we obtain $s(\frac{2}{5}) = (5, 1, 3, 3, 1, 5, 4, 1, 3, 4)$.

The right endpoint of $\overline{L_t}$ lies in the interior of the upper-rightmost edge of $\widetilde{\mathbb{R}^2}$, and hence no sign is assigned to this edge. Notice also that the only horizontal edge crossed by $\overline{L_t}$ appears to

[1] In [Aig13], the term "strongly admissible sequence" is used for the bi-infinite sequence ${}^{\infty}s(t)^{\infty}$ in the sense of that paper. Here we use the same terminology for the single block $s(t)$.

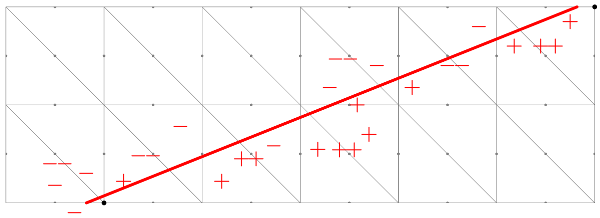


FIGURE 3. The segment $\overline{L_t}$ corresponding to $t = \frac{2}{5}$ and its signs

be crossed at its midpoint; in fact, however, $\overline{L_t}$ is defined by shifting the line slightly to the left (more precisely, this shift has already been made at the stage of the line L_t before constructing $\overline{L_t}$). Thus the intersection point lies slightly to the left of the midpoint. Consequently the corresponding sign is $(+)$.

REMARK 7.4.3. (0) n is odd.
(1) $a_0 = 2 + k_1 + k_2 + k_3$.
(2) If $t = \frac{1}{1}$, then $s(t) = (2 + k_1 + k_2 + k_3, 2 + k_{\sigma(2)})$.
(3) If $t \in (0, 1)$, then $a_1 = 1$ and $a_n \neq 1$. If $t \in (1, \infty)$, then $a_1 \neq 1$ and $a_n = 1$.
(4) If $t = \frac{1}{2}$, then $n = 3$ and $a_3 = a_2 + 1 + k_t$. If $t = \frac{2}{1}$, then $n = 3$ and $a_1 = a_2 + 1 + k_t$.
(5) If $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, then $a_2 + 1 = a_n$ and $a_{2+i} = a_{n-i}$ for $i = 1, 2, \dots, \frac{n-5}{2}$; when $n = 5$ there is no such relation. Moreover $a_{\frac{n+3}{2}} = a_{\frac{n+1}{2}} + (-1)^{\frac{n+1}{2}} k_t$.
(6) If $t \in (1, \frac{2}{1}) \cup (\frac{2}{1}, \infty)$, then $a_2 = a_{n-1} + 1$ and $a_{1+i} = a_{n-i-1}$ for $i = 1, 2, \dots, \frac{n-5}{2}$; when $n = 5$ there is no such relation. Moreover $a_{\frac{n+1}{2}} = a_{\frac{n-1}{2}} + (-1)^{\frac{n-1}{2}} k_t$.
(7) For $t \in (0, \infty)$, let $s^*(\frac{1}{t})$ denote the $(k_1, k_2, k_3, \sigma^*)$-generalized strongly admissible sequence attached to $\frac{1}{t}$. Then it is given by $(a_0, a_n, a_{n-1}, \dots, a_1)$. This follows by reflecting L_t across the line of slope 1 through the origin and applying the symmetries in (3)–(6). Clearly ${}^\infty(s^*(\frac{1}{t}))^\infty$ is the reverse of ${}^\infty s(t)^\infty$. The same statement also holds for $t = \frac{0}{1}$ and $t = \frac{1}{0}$.

The main theorem of this section is the following.

THEOREM 7.4.4. *For every reduced fraction $t \in (0, \infty]$, $C_t = F_{s(t)}$.*

Before proving this theorem, we observe that Theorem 6.3.17, postponed at the end of Section 3, follows immediately from it.

Theorem 6.3.17 (recalled). Fix $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$, and let p and q be coprime nonnegative integers. If $A = (0, 0)$ and $B = (q, p)$, then $|\gamma^R_{AB}| = |\gamma^L_{AB}| = m_{\frac{p}{q}}$. In particular, $d(A, B) = m_{\frac{p}{q}}$.

PROOF. Let $s(t) = (a_0, \dots, a_n)$. Combining Corollary 6.1.8, which gives the entries of $F_{s(t)}$, with Theorems 7.2.1 and 7.4.4, we obtain

$$\begin{bmatrix} Km_t - k_t - u_t & \frac{Km_t u_t - k_t u_t - u_t^2 - 1}{m_t} \\ m_t & u_t \end{bmatrix} = \begin{bmatrix} N(a_0, \dots, a_n) & N(a_0, \dots, a_{n-1}) \\ N(a_1, \dots, a_n) & N(a_1, \dots, a_{n-1}) \end{bmatrix}.$$

Comparing the $(2, 1)$ entries gives $m_t = N(a_1, \dots, a_n)$. By the construction of $\overline{L_t}$, the latter number is the GM length of $L_t (= \gamma^L_{AB})$. The equality for γ^R_{AB} follows from the same argument, or equivalently from the symmetry between the two perturbations. Hence the assertion follows. □

To prove the theorem, we first work in the range $t \in (0, 1)$. For $s(t) = (a_0, \dots, a_n)$, set $s'_-(t) := (a_2 + 1, a_3, \dots, a_n)$.

PROPOSITION 7.4.5. *Let (r, t, s) be a Farey triple with $t \in (0, 1)$. Then the following three assertions hold.*

(1) *If* $r=\frac{0}{1}$, $s\neq\frac{1}{1}$, *and* $s'_{-}(s)=(b_1,\dots,b_m)$, *then*
$$s'_{-}(t)=(b_m,b_{m-1},\dots,b_1-1,1,2+k_{\sigma(2)}+k_{\sigma(3)}).$$
(2) *If* $r\neq\frac{0}{1}$, $s=\frac{1}{1}$, *and* $s'_{-}(r)=(a_1,\dots,a_\ell)$, *then*
$$s'_{-}(t)=(2+k_{\sigma(2)},2+k_1+k_2+k_3,a_\ell,\dots,a_1).$$
(3) *If* $r\neq\frac{0}{1}$, $s\neq\frac{1}{1}$, $s'_{-}(r)=(a_1,\dots,a_\ell)$, *and* $s'_{-}(s)=(b_1,\dots,b_m)$, *then*
$$s'_{-}(t)=(b_m,\dots,b_1-1,1,2+k_1+k_2+k_3,a_\ell,\dots,a_1).$$

Proof. Let $\mathcal{PG}(t)$ denote the union of all triangles in $\widetilde{\mathbb{R}^2}$ crossed by L_t.

We first prove (1). Under the assumption $r=\frac{0}{1}$ and $s\neq\frac{1}{1}$, there exists $p\in\mathbb{Z}_{>1}$ such that $s=\frac{1}{p}$, and it is enough to prove the assertion in the case $t=\frac{1}{p+1}$. Since $p\geq 2$, the last $2+k_{\sigma(2)}+k_{\sigma(3)}$ signs in $\mathcal{PG}(\frac{1}{p+1})$ are all $+$, and the sign immediately preceding them is $-$; see Figure 4.

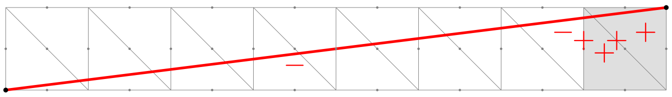


Figure 4. The graph $\mathcal{PG}(\frac{1}{p+1})$ for $k=1$ and $p=7$

We compare the sign sequence obtained from $\mathcal{PG}(\frac{1}{p+1})$ after removing the final square consisting of the last two right triangles with the sign sequence of $\mathcal{PG}(\frac{1}{p})$; compare the white part of Figure 4 with Figure 5. We denote the former graph by $\mathcal{SPG}(\frac{1}{p+1})$.

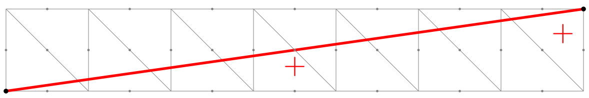


Figure 5. The graph $\mathcal{PG}(\frac{1}{p})$ for $k=1$ and $p=7$

We show that $\mathcal{SPG}(\frac{1}{p+1})$ and $\mathcal{PG}(\frac{1}{p})$ differ only in the signs corresponding to the central edge and the last triangle. It is clear that these two signs are different. Thus it remains to prove that all other signs agree. First, the signs assigned to the right triangles clearly agree. Next consider the signs assigned to vertical edges. The height of the intersection of the a-th vertical edge from the left in $\mathcal{PG}(\frac{1}{p})$ with the segment $L_{\frac{1}{p}}$ is $\frac{a}{p}$. The height of the intersection of the a-th vertical edge from the left in $\mathcal{SPG}(\frac{1}{p+1})$ with the segment $L_{\frac{1}{p+1}}$ is $\frac{a}{p+1}$. Since $\frac{a}{p}>\frac{a}{p+1}$, it is immediate that if $\frac{a}{p}<\frac{1}{2}$, then $\frac{a}{p+1}<\frac{1}{2}$. Hence every $(-)$ sign in $\mathcal{PG}(\frac{1}{p})$ is preserved in $\mathcal{SPG}(\frac{1}{p+1})$. It is therefore enough to prove preservation of $(+)$ signs, namely that if $\frac{a}{p}>\frac{1}{2}$, then $\frac{a}{p+1}\geq\frac{1}{2}$. From the assumption, $2a\geq p+1$, and hence
$$\frac{1}{2}-\frac{a}{p+1}=\frac{p-2a+1}{2(p+1)}\leq 0.$$
This gives the desired inequality. The signs assigned to diagonal edges are treated in the same way. Therefore $\mathcal{SPG}(\frac{1}{p+1})$ and $\mathcal{PG}(\frac{1}{p})$ differ only in the signs corresponding to the central edge and the last triangle. At this point, by the symmetries described in Remark 7.4.3 (4), (5), changing the central sign of $\mathcal{PG}(\frac{1}{p})$ reverses the sequence itself. Thus the sequence constructed from the sign sequence of $\mathcal{SPG}(\frac{1}{p+1})$ is $(b_m,b_{m-1},\dots,b_1-1,1)$. Combining this with the value obtained from the final tile of $\mathcal{PG}(\frac{1}{p+1})$, namely the block of $2+k_{\sigma(2)}+k_{\sigma(3)}$ signs, proves the assertion.

We next prove (2). Under the assumption $r\neq\frac{0}{1}$ and $s=\frac{1}{1}$, there exists $p\in\mathbb{Z}_{>1}$ such that $r=\frac{p}{p+1}$, and it is enough to prove the assertion in the case $t=\frac{p+1}{p+2}$. Since $p\geq 2$, the first

$4 + k_{\sigma(1)} + 2k_{\sigma(2)} + k_{\sigma(3)}$ signs in $\mathcal{PG}(\frac{p+1}{p+2})$ consist of $2 + k_{\sigma(2)}$ signs $-$ and $2 + k_1 + k_2 + k_3$ signs $+$; see Figure 6.

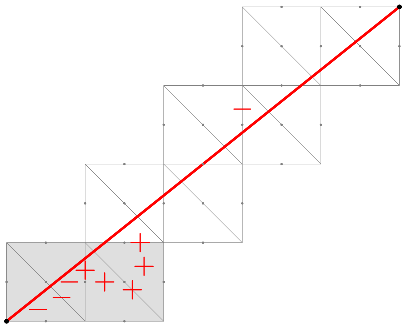


FIGURE 6. The graph $\mathcal{PG}(\frac{p+1}{p+2})$ for $k = 1$ and $p = 3$

We compare the sign sequence obtained from $\mathcal{PG}(\frac{p+1}{p+2})$ after removing the first two tiles with the sign sequence of $\mathcal{PG}(\frac{p}{p+1})$; compare Figure 6 with Figure 7. We denote the former graph by $\mathcal{SPG}(\frac{p+1}{p+2})$.

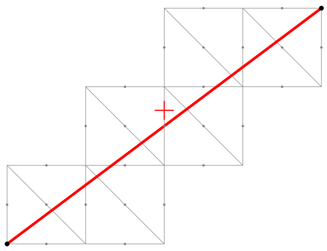


FIGURE 7. The graph $\mathcal{PG}(\frac{p}{p+1})$ for $k = 1$ and $p = 3$

We show that $\mathcal{SPG}(\frac{p+1}{p+2})$ and $\mathcal{PG}(\frac{p}{p+1})$ differ only in the sign corresponding to the central edge. It is clear that the sign at this position is different. Thus it remains to prove that all other signs agree. First, the signs assigned to the right triangles clearly agree. The height of the intersection of the $(a+1)$-st vertical edge from the left in $\mathcal{PG}(\frac{p}{p+1})$ with the segment $L_{\frac{p}{p+1}}$ is $\frac{ap}{p+1}$, whereas the height of the intersection of the $(a+1)$-st vertical edge from the left in $\mathcal{SPG}(\frac{p+1}{p+2})$ with the segment $L_{\frac{p+1}{p+2}}$ is $\frac{(a+1)(p+1)}{p+2} - 1$ (where the origin is taken to be the lower-left point of $\mathcal{SPG}(\frac{p+1}{p+2})$). Therefore it is enough to prove the implication

$$\frac{ap}{p+1} - (a-1) > \frac{1}{2} \quad \Longrightarrow \quad \frac{(a+1)(p+1)}{p+2} - 1 - (a-1) \geq \frac{1}{2}.$$

Indeed, from $2a \leq p$, we obtain

$$\frac{1}{2} - \frac{(a+1)(p+1)}{p+2} + 1 + (a-1) = \frac{2a-p}{2(p+2)} \leq 0,$$

and the desired inequality follows. The signs assigned to diagonal and horizontal edges are treated in the same way. The rest of the argument is the same as in case (1). □

For the proof of the remaining case, we recall the upper Christoffel word. Let a/b be a reduced fraction with $b > 0$. For $1 \leq i \leq b+1$, let y_i be the height of the intersection of $L_{a/b}$ with the i-th vertical line, counted from the left, in $\mathcal{PG}(a/b)$. Define

$$\mathrm{uch}_{a/b} := w_1 \cdots w_b \in \{X, Y\}^*$$

by

$$w_i = \begin{cases} X & \text{if } \lceil y_{i+1} \rceil - \lceil y_i \rceil = 1, \\ Y & \text{if } \lceil y_{i+1} \rceil - \lceil y_i \rceil = 0. \end{cases}$$

EXAMPLE 7.4.6. For example, the upper Christoffel word $\mathrm{uch}_{2/5}$ is $XYXYY$; see also Figure 8.

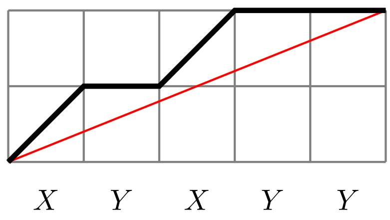


FIGURE 8. The upper Christoffel word $\mathrm{uch}_{2/5}$

THEOREM 7.4.7. *Let $(r, t, s) \in \mathrm{F}\mathbb{T}$, and assume that $t \neq \frac{1}{0}$. Then*

$$\mathrm{uch}_t = \mathrm{uch}_s \cdot \mathrm{uch}_r$$

holds. Here $\cdot$ denotes concatenation of words.

PROOF. Write $r = \frac{a}{b}$ and $s = \frac{c}{d}$ in lowest terms. Since $(r, t, s) \in \mathrm{F}\mathbb{T}$, we have $r < t < s$ and $bc - ad = 1$. Hence $t = \frac{a+c}{b+d}$. By the definition of the upper Christoffel word, if we put

$$\Delta_i(u) := \lceil iu \rceil - \lceil (i-1)u \rceil ,$$

then the i-th letter of uch_u is determined by $\Delta_i(u)$.

We first prove that $\Delta_i(t) = \Delta_i(s)$ for $1 \leq i \leq d$. Since

$$s - t = \frac{c}{d} - \frac{a+c}{b+d} = \frac{bc - ad}{d(b+d)} = \frac{1}{d(b+d)},$$

we have, for $1 \leq i \leq d-1$,

$$0 < is - it < \frac{1}{d}.$$

On the other hand, since $\gcd(c, d) = 1$, the number is is not an integer, and its fractional part is at least $1/d$. Therefore

$$\lceil it \rceil = \lceil is \rceil \qquad (1 \leq i \leq d-1).$$

Moreover, since

$$dt = c - \frac{1}{b+d},$$

we have $\lceil dt \rceil = c = \lceil ds \rceil$. Hence

$$\Delta_i(t) = \Delta_i(s) \qquad (1 \leq i \leq d)$$

follows.

Next we prove that $\Delta_{d+j}(t) = \Delta_j(r)$ for $1 \leq j \leq b$. We have

$$(d+j)t - c = \frac{(d+j)(a+c)}{b+d} - c = jr + \frac{j-b}{b(b+d)}.$$

Thus, for $1 \leq j \leq b-1$,

$$0 < jr - ((d+j)t - c) < \frac{1}{b}.$$

Since $\gcd(a, b) = 1$, the number jr is not an integer, and its fractional part is at least $1/b$. Hence

$$\lceil (d+j)t \rceil - c = \lceil jr \rceil \qquad (1 \leq j \leq b-1).$$

The case $j = b$ is obtained by direct computation in the same way. Therefore

$$\Delta_{d+j}(t) = \lceil (d+j)t \rceil - \lceil (d+j-1)t \rceil = \lceil jr \rceil - \lceil (j-1)r \rceil = \Delta_j(r)$$

follows. Taking into account that the word uch_t has length $b+d$, we obtain

$$\mathrm{uch}_t = (\text{the part of length } d \text{ coming from } \mathrm{uch}_s) \cdot (\text{the part of length } b \text{ coming from } \mathrm{uch}_r).$$

Thus $\mathrm{uch}_t = \mathrm{uch}_s \cdot \mathrm{uch}_r$. □

The preceding theorem gives the following decomposition of $\mathcal{PG}(t)$ in the case needed below.

Corollary 7.4.8. *Let $(r,t,s) \in \mathbb{FT}$ with $t \in (0,1)$. Then $\mathcal{PG}(t)$ decomposes, from the lower-left side to the upper-right side, into $\mathcal{PG}(s)$, one intermediate tile, and $\mathcal{PG}(r)$.*

Proof of Proposition 7.4.5 (3). By Corollary 7.4.8, the graph $\mathcal{PG}(t)$ decomposes into $\mathcal{PG}(s)$, one intermediate tile, and $\mathcal{PG}(r)$; notice that the order of the r- and s-parts is reversed, as in Figure 9. We denote by $\mathcal{SPG}(s)$ the part of $\mathcal{PG}(t)$ corresponding to $\mathcal{PG}(s)$, and by $\mathcal{SPG}(r)$ the part corresponding to $\mathcal{PG}(r)$. In this decomposition, every sign attached to the intermediate tile is $+$.

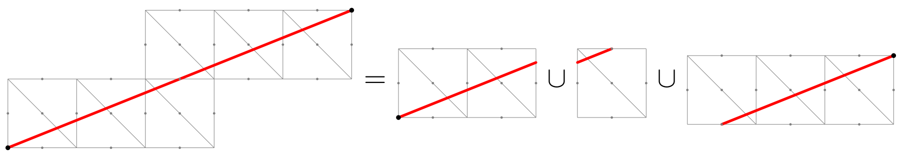


Figure 9. Decomposition of $\mathcal{PG}(t)$ for $r = \frac{1}{3}$, $t = \frac{2}{5}$, and $s = \frac{1}{2}$

Write $r = a/b$ and $s = c/d$ in lowest terms. The distance from the upper-rightmost point of $\mathcal{SPG}(s)$ to the intersection of L_t with the vertical edge having this point as an endpoint is $\frac{1}{b+d}$. Similarly, the distance from the lower-leftmost point of $\mathcal{SPG}(r)$ to the intersection of L_t with the horizontal edge having this point as an endpoint is $\frac{1}{a+c}$. When the sign sequence of $\mathcal{SPG}(s)$ is compared with that of $\mathcal{PG}(s)$, the midpoint comparisons used in the proof of (1) show that all signs agree except possibly the sign of the last triangle and the sign of the central edge. The displacement at the end of the s-part is exactly the amount $1/(b+d)$ displayed above, so these two exceptional positions are precisely the ones that produce the terminal change from b_1 to $b_1 - 1$ and then the additional block of length 1.

Next compare the sign sequence of $\mathcal{SPG}(r)$ with that of $\mathcal{PG}(r)$. The midpoint comparisons used in the proof of (2), now with the horizontal displacement $1/(a+c)$ at the beginning of the r-part, show that all signs agree except at the central edge. Thus the r-part contributes the block $(a_\ell, \dots, a_1)$ unchanged. The intermediate tile contributes the block of $+$ signs of length $2 + k_1 + k_2 + k_3$. Combining the s-part, the intermediate tile, and the r-part gives

$$s'_{-}(t) = (b_m, \dots, b_1 - 1, 1, 2 + k_1 + k_2 + k_3, a_\ell, \dots, a_1),$$

which is the desired formula. □

We now rephrase the statement above in the form that will be used in the induction below. If $u \in (0,1)$ and $s(u) = (K-1, c_1, \dots, c_q)$, then we write $s_{-}(u) := (c_1, \dots, c_q)$. Thus $s_{-}(u)$ is obtained from $s(u)$ by deleting its first entry.

Corollary 7.4.9. *Let (r,t,s) be a Farey triple with $t \in (0,1)$.*

(1) *If $r = \frac{0}{1}$, $s \neq \frac{1}{1}$, and $s_{-}(s) = (b_1, \dots, b_m)$, then*

$$s_{-}(t) = (1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}).$$

(2) *If $r \neq \frac{0}{1}$, $s = \frac{1}{1}$, and $s_{-}(r) = (a_1, \dots, a_\ell)$, then*

$$s_{-}(t) = (1, 1 + k_{\sigma(2)}, 2 + k_1 + k_2 + k_3, a_\ell, \dots, a_3, a_2 + 1).$$

(3) *If $r \neq \frac{0}{1}$, $s \neq \frac{1}{1}$, $s_{-}(r) = (a_1, \dots, a_\ell)$, and $s_{-}(s) = (b_1, \dots, b_m)$, then*

$$s_{-}(t) = (1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_1 + k_2 + k_3, a_\ell, \dots, a_3, a_2 + 1).$$

We record relations for the numbers of order ideals that will be needed in the proof. For any finite integer sequence $S = (x_1, \dots, x_m)$, define

$$F_S := \prod_{i=1}^{m} \begin{bmatrix} x_i & 1 \\ 1 & 0 \end{bmatrix}, \qquad N(S) := (F_S)_{11}.$$

To use the recursion at the endpoints, we set $N(\emptyset) = 1$ and agree that terms corresponding to sequences of length -1 are 0.

LEMMA 7.4.10. *Let $a, b, k, a_1, \dots, a_n \in \mathbb{Z}$, and let μ be an integer sequence. Then the following identities hold; in (2) assume $n \geq 2$.*

(1) $N(a, 1, b, \mu) = (a + 1)N(b + 1, \mu) - N(\mu)$.
(2) $N(a_2, \dots, a_n + k, a_n, \dots, a_1) = N(a_1, \dots, a_n + k, a_n, \dots, a_2) + (-1)^n k$.

PROOF. For a finite integer sequence $S = (x_1, \dots, x_m)$, put $F_x := [\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}]$, $F_S := F_{x_1} \cdots F_{x_m}$, and $N(S) = (F_S)_{11}$. We first record two elementary identities that will be used in the proof.
Claim 1. For any finite integer sequence $(x_1, \dots, x_m)$, under the convention on the empty sequence and sequences of length -1, we have

$$N(x_1, \dots, x_m) = x_1 N(x_2, \dots, x_m) + N(x_3, \dots, x_m). \tag{7.4.1}$$

Claim 2. For any integer c and integer sequence ν, we have

$$N(c + 1, \nu) = N(c, \nu) + N(\nu). \tag{7.4.2}$$

Indeed, applying (7.4.1) with $x_1 = c + 1$ gives $N(c + 1, \nu) = (c + 1)N(\nu) + N(\nu')$, where ν' denotes the sequence obtained from ν by deleting its first term. Similarly, $N(c, \nu) = cN(\nu) + N(\nu')$. Subtracting these two equalities gives (7.4.2). We now prove (1) and (2).

We first prove (1). Using (7.4.1) and Proposition 6.1.9, we have $N(a, 1, b, \mu) = aN(b+1, \mu) + N(b, \mu)$ and $N(b, \mu) = N(b + 1, \mu) - N(\mu)$. Therefore

$$\begin{aligned} N(a, 1, b, \mu) &= aN(b + 1, \mu) + N(b, \mu) \\ &= aN(b + 1, \mu) + \big(N(b + 1, \mu) - N(\mu)\big) \\ &= (a + 1)N(b + 1, \mu) - N(\mu). \end{aligned}$$

Thus (1) has been proved.

We next prove (2). Put $P := F_{a_2} \cdots F_{a_{n-1}} = [\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}]$, and write $A_j := F_{a_j}$. Since each F_x is symmetric, $P^T = F_{a_{n-1}} \cdots F_{a_2}$. Put $E_{11} := [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$. Then $F_{a_n+k} = A_n + kE_{11}$.

$$\Delta(k) := N(a_2, \dots, a_n + k, a_n, \dots, a_1) - N(a_1, \dots, a_n + k, a_n, \dots, a_2)$$

$$\begin{aligned} \Delta(k) &= \big(PF_{a_n+k}A_nP^TA_1\big)_{11} - \big(A_1PF_{a_n+k}A_nP^T\big)_{11} \\ &= \big(P(A_n + kE_{11})A_nP^TA_1\big)_{11} - \big(A_1P(A_n + kE_{11})A_nP^T\big)_{11}. \end{aligned}$$

Expanding by distributivity, we obtain

$$\begin{aligned} \Delta(k) = &\Big(\big(PA_nA_nP^TA_1\big)_{11} - \big(A_1PA_nA_nP^T\big)_{11}\Big) \\ &+ k\Big(\big(PE_{11}A_nP^TA_1\big)_{11} - \big(A_1PE_{11}A_nP^T\big)_{11}\Big). \end{aligned}$$

The two sequences

$$(a_2, \dots, a_n, a_n, \dots, a_1) \quad \text{and} \quad (a_1, \dots, a_n, a_n, \dots, a_2)$$

are reverses of one another. Hence Proposition 6.1.9 (1) gives $\Delta(0) = 0$. Therefore

$$\Delta(k) = k\Big(\big(PE_{11}A_nP^TA_1\big)_{11} - \big(A_1PE_{11}A_nP^T\big)_{11}\Big). \tag{7.4.3}$$

We now compute the expression in parentheses. First,

$$PE_{11} = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix}.$$

Therefore

$$PE_{11}A_n = \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} pa_n & p \\ qa_n & q \end{bmatrix}.$$

Moreover, $P^T = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$. Hence

$$PE_{11}A_nP^T = \begin{bmatrix} pa_n & p \\ qa_n & q \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p(a_np+r) & p(a_nq+s) \\ q(a_np+r) & q(a_nq+s) \end{bmatrix}.$$

Thus

$$(PE_{11}A_nP^TA_1)_{11} = \begin{bmatrix} p(a_np+r) & p(a_nq+s) \end{bmatrix} \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = a_1p(a_np+r) + p(a_nq+s),$$

Moreover

$$(A_1PE_{11}A_nP^T)_{11} = \begin{bmatrix} a_1 & 1 \end{bmatrix} \begin{bmatrix} p(a_np+r) \\ q(a_np+r) \end{bmatrix} = a_1p(a_np+r) + q(a_np+r).$$

Therefore

$$\begin{aligned}
&(PE_{11}A_nP^TA_1)_{11} - (A_1PE_{11}A_nP^T)_{11} \\
&\quad = (a_1p(a_np+r) + p(a_nq+s)) - (a_1p(a_np+r) + q(a_np+r)) \\
&\quad = p(a_nq+s) - q(a_np+r) \\
&\quad = pa_nq + ps - qa_np - qr \\
&\quad = ps - qr = \det P.
\end{aligned}$$

Here $P = F_{a_2} \cdots F_{a_{n-1}}$ is a product of $n-2$ matrices, each of determinant -1. Hence $\det P = (-1)^{n-2} = (-1)^n$. Substituting this into (7.4.3) gives $\Delta(k) = (-1)^n k$. By the definition of $\Delta(k)$, this is precisely

$$N(a_2, \ldots, a_n + k, a_n, \ldots, a_1) = N(a_1, \ldots, a_n + k, a_n, \ldots, a_2) + (-1)^n k.$$

□

LEMMA 7.4.11. *Let $\emptyset$ denote the empty sequence, and set*

$$N(\emptyset) = 1, \qquad N(\emptyset, \lambda) = N(\lambda).$$

We use this convention throughout. Then

$$N(\nu, a, 1, b, \mu) = N(\nu, a+1)N(b+1, \mu) - N(\nu)N(\mu).$$

PROOF. We prove this by induction on the length of ν. If $\nu = \emptyset$, the assertion becomes

$$N(a, 1, b, \mu) = (a+1)N(b+1, \mu) - N(\mu),$$

which is Lemma 7.4.10 (1).

For the induction step, write $\nu = (c, \nu')$. Let ν'' be the sequence obtained from ν' by deleting its first term; if $|\nu| = 1$, then $\nu' = \nu'' = \emptyset$.

$$N(\nu, a, 1, b, \mu) = c\, N(\nu', a, 1, b, \mu) + N(\nu'', a, 1, b, \mu).$$

By the induction hypothesis,

$$\begin{aligned}
&N(\nu, a, 1, b, \mu) \\
&= c\Big(N(\nu', a+1)N(b+1, \mu) - N(\nu')N(\mu)\Big) + \Big(N(\nu'', a+1)N(b+1, \mu) - N(\nu'')N(\mu)\Big) \\
&= \Big(c\, N(\nu', a+1) + N(\nu'', a+1)\Big)N(b+1, \mu) - \Big(c\, N(\nu') + N(\nu'')\Big)N(\mu).
\end{aligned}$$

Using the recurrence for N again,

$$c\, N(\nu', a+1) + N(\nu'', a+1) = N(\nu, a+1), \quad c\, N(\nu') + N(\nu'') = N(\nu).$$

Hence

$$N(\nu, a, 1, b, \mu) = N(\nu, a+1)N(b+1, \mu) - N(\nu)N(\mu).$$

This completes the induction. □

LEMMA 7.4.12. *Let $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_s)$ be nonempty finite sequences of positive integers. Write (α, β) for concatenation. Let α^- denote the sequence obtained by deleting the last term of α, let ${}^-\alpha$ denote the sequence obtained by deleting the first term, and let ${}^-\alpha^-$ denote the sequence obtained by deleting both the first and last terms; define the analogous notation for β. We set $N(\emptyset) = 1$ and agree that a term corresponding to a sequence of length -1 is 0. Then*

$$N(\alpha, \beta) = N(\alpha)N(\beta) + N(\alpha^-)N({}^-\beta).$$

PROOF. By Corollary 6.1.8,

$$F_\alpha = \begin{bmatrix} N(\alpha) & N(\alpha^-) \\ N({}^-\alpha) & N({}^-\alpha^-) \end{bmatrix}, \qquad F_\beta = \begin{bmatrix} N(\beta) & N(\beta^-) \\ N({}^-\beta) & N({}^-\beta^-) \end{bmatrix}.$$

By the multiplicativity of the matrices, $F_{\alpha,\beta} = F_\alpha F_\beta$. Hence

$$F_{\alpha,\beta} = \begin{bmatrix} N(\alpha) & N(\alpha^-) \\ N({}^-\alpha) & N({}^-\alpha^-) \end{bmatrix} \begin{bmatrix} N(\beta) & N(\beta^-) \\ N({}^-\beta) & N({}^-\beta^-) \end{bmatrix}.$$

The $(1,1)$ entry of the right-hand side is $N(\alpha)N(\beta) + N(\alpha^-)N({}^-\beta)$. On the other hand, by Corollary 6.1.8, the $(1,1)$ entry of $F_{\alpha,\beta}$ is $N(\alpha, \beta)$. Comparing the $(1,1)$ entries gives

$$N(\alpha, \beta) = N(\alpha)N(\beta) + N(\alpha^-)N({}^-\beta).$$

□

We now prove Theorem 7.4.4. For $t \in (0, 1]$ we use induction; the remaining cases are reduced to this range.

PROOF OF THEOREM 7.4.4. First assume $t \in (0, 1]$ and prove the assertion in this range. The cases $t = \frac{1}{1}$ and $t = \frac{1}{2}$ are checked by direct computation.

We first assume that the theorem holds for $t = \frac{1}{p-1}$ and $t = \frac{1}{p}$, where $p \geq 2$, and prove it for $t = \frac{1}{p+1}$. By Corollary 7.4.9, we may write

$$s(\tfrac{1}{p}) = (2 + k_1 + k_2 + k_3, b_1, b_2, \dots, b_m),$$

and hence

$$s(\tfrac{1}{p+1}) = (2 + k_1 + k_2 + k_3, 1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}).$$

Therefore,

$$F_{s(\frac{1}{p+1})} = \begin{bmatrix} N(K-1, 1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}) & N(K-1, 1, b_m - 1, b_{m-1}, \dots, b_1) \\ N(1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}) & N(1, b_m - 1, b_{m-1}, \dots, b_1) \end{bmatrix}.$$

On the other hand, since $C_{\frac{1}{p+1}} = C_{\frac{0}{1}} C_{\frac{1}{p}} - D_{\frac{1}{p-1}}$, we compare the entries of $F_{s(\frac{1}{p+1})}$ with those of $C_{\frac{1}{p+1}}$ after applying the induction hypothesis to $C_{\frac{1}{p}}$. Since $\det(C_{\frac{1}{p+1}}) = \det(F_{s(\frac{1}{p+1})}) = 1$ by Lemma 2.2.5, Remark 7.4.3 (0), and Theorem 7.2.1, it is enough to compare the $(1,1)$, $(2,1)$, and $(1,2)$ entries. Thus it remains to prove the following three identities:

$$\begin{aligned} &N(K-1, 1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}) \\ &\quad = KN(K-1, b_1, \dots, b_m) - Kk_{\sigma(1)}N(b_1, \dots, b_m) - N(b_1, \dots, b_m) - k_{\frac{1}{p-1}}, \end{aligned} \tag{7.4.4}$$

$$\begin{aligned} &N(1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}) \\ &\quad = N(K-1, 1, b_2 - 1, \dots, b_m) - k_{\sigma(1)}N(b_1, \dots, b_m), \end{aligned} \tag{7.4.5}$$

$$\begin{aligned} &N(1, b_m - 1, b_{m-1}, \dots, b_1) \\ &\quad = N(K-1, 1, b_2 - 1, \dots, b_{m-1}) - k_{\sigma(1)}N(b_1, \dots, b_{m-1}) - k_{\frac{1}{p-1}}. \end{aligned} \tag{7.4.6}$$

We prove (7.4.4). Applying Lemma 7.4.10 (1) to the left-hand side gives

$$\begin{aligned} &N(K-1, 1, b_m - 1, b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}) \\ &= KN(b_m, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}) - N(b_{m-1}, \dots, b_1, 2 + k_{\sigma(2)} + k_{\sigma(3)}). \end{aligned}$$

Since $b_1 = 1$, we reverse the sequence by Proposition 6.1.9 (1) and then apply Lemma 7.4.10 (1). This gives

$$\begin{aligned}&KN(b_m,\dots,b_1,2+k_{\sigma(2)}+k_{\sigma(3)}) - N(b_{m-1},\dots,b_1,2+k_{\sigma(2)}+k_{\sigma(3)})\\&= (K-k_{\sigma(1)})KN(b_2+1,\dots,b_m) - KN(b_3,\dots,b_m) - (K-k_{\sigma(1)})N(b_2+1,\dots,b_{m-1})\\&\quad + N(b_3,\dots,b_{m-1}).\end{aligned}$$

On the other hand, applying Lemma 7.4.10 (1) to the first term on the right-hand side of (7.4.4), we obtain

$$\begin{aligned}&KN(K-1,b_1,\dots,b_m) - Kk_{\sigma(1)}N(b_1,\dots,b_m) - N(b_1,\dots,b_m) - k_{\frac{1}{p-1}}\\&= K^2N(b_2+1,\dots,b_m) - KN(b_3,\dots,b_m) - Kk_{\sigma(1)}N(b_1,\dots,b_m) - N(b_1,\dots,b_m) - k_{\frac{1}{p-1}}.\end{aligned}$$

Since Corollary 7.4.9 applied to $r = \frac{0}{1}$, $t = \frac{1}{p}$, and $s = \frac{1}{p-1}$ gives $b_{m-1} = 1$ and $b_m = 2+k_{\sigma(2)}+k_{\sigma(3)}$, applying Lemma 7.4.10 (1) to the fourth term in the preceding expression yields

$$\begin{aligned}&K^2N(b_2+1,\dots,b_m) - KN(b_3,\dots,b_m) - Kk_{\sigma(1)}N(b_1,\dots,b_m) - N(b_1,\dots,b_m) - k_{\frac{1}{p-1}}\\&= K^2N(b_2+1,\dots,b_m) - KN(b_3,\dots,b_m) - Kk_{\sigma(1)}N(b_1,\dots,b_m)\\&\qquad - (K-k_{\sigma(1)})N(b_1,\dots,b_{m-2}+1) + N(b_1,\dots,b_{m-3}) - k_{\frac{1}{p-1}}.\end{aligned}$$

Using $N(b_1,\dots,b_{m-2}+1) = N(b_1,\dots,b_{m-1})$ and comparing the expression obtained from the left-hand side with the expression obtained from the right-hand side, it remains to show that the difference is zero. This difference is

$$E = N(b_3,\dots,b_{m-2}+1) - N(b_2+1,\dots,b_{m-3}) + k_{\frac{1}{p-1}}.$$

Taking the symmetry in Remark 7.4.3 (5) into account, Lemma 7.4.10 (2) gives $E = 0$. Hence (7.4.4) follows. The identities (7.4.5) and (7.4.6) are proved by the same transformations.

We next assume that the theorem holds for $t = \frac{p-1}{p}$ and $t = \frac{p}{p+1}$, and prove it for $t = \frac{p+1}{p+2}$. By Corollary 7.4.9, we may write

$$s(\tfrac{p}{p+1}) = (2+k_1+k_2+k_3, a_1, a_2, \dots, a_\ell),$$

and hence

$$s(\tfrac{p+1}{p+2}) = (2+k_1+k_2+k_3, 1, 1+k_{\sigma(2)}, 2+k_1+k_2+k_3, a_\ell, \dots, a_3, a_2+1).$$

Therefore,

$$F_{s(\frac{p+1}{p+2})} = \begin{bmatrix} N(K-1,1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_2+1) & N(K-1,1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_3) \\ N(1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_2+1) & N(1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_3) \end{bmatrix}.$$

On the other hand, since $C_{\frac{p+1}{p+2}} = C_{\frac{p}{p+1}}C_{\frac{1}{1}} - D_{\frac{p-1}{p}}$, we compare the entries of $F_{s(\frac{p+1}{p+2})}$ with those of $C_{\frac{p+1}{p+2}}$ after applying the induction hypothesis. For the same reason as in the preceding case, it is enough to compare the $(1,1)$, $(2,1)$, and $(1,2)$ entries. Thus it remains to prove the following three identities:

$$\begin{aligned}&N(K-1,1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_2+1)\\&\quad= (K(k_{\sigma(2)}+2)-k_{\sigma(2)}-1)N(K-1,a_1,\dots,a_\ell)\\&\qquad+ (k_{\sigma(2)}+2)N(K-1,a_1,\dots,a_{\ell-1}) - k_{\frac{p-1}{p}},\end{aligned}\tag{7.4.7}$$

$$\begin{aligned}&N(1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_2+1)\\&\quad= (K(k_{\sigma(2)}+2)-k_{\sigma(2)}-1)N(a_1,\dots,a_\ell)\\&\qquad+ (k_{\sigma(2)}+2)N(a_1,\dots,a_{\ell-1}),\end{aligned}\tag{7.4.8}$$

$$\begin{aligned}&N(1,1+k_{\sigma(2)},K-1,a_\ell,\dots,a_3)\\&\quad= (K-1)N(a_1,\dots,a_\ell) + N(a_1,\dots,a_{\ell-1}) - k_{\frac{p-1}{p}}.\end{aligned}\tag{7.4.9}$$

First note that $a_1 = 1$ by Remark 7.4.3 (3). Therefore Proposition 6.1.9 gives

$$N(a_\ell, \dots, a_3, a_2+1) = N(a_2+1, a_3, \dots, a_\ell) = N(1, a_2, \dots, a_\ell) = N(a_1, \dots, a_\ell),$$
$$N(a_{\ell-1}, \dots, a_3, a_2+1) = N(a_2+1, a_3, \dots, a_{\ell-1}) = N(1, a_2, \dots, a_{\ell-1}) = N(a_1, \dots, a_{\ell-1}).$$

We will use this identification below. We first prove (7.4.7). By Lemma 7.4.10 (1), the left-hand side is

$$\begin{aligned}&N(K-1, 1, 1+k_{\sigma(2)}, K-1, a_\ell, \dots, a_3, a_2+1)\\ &= KN(k_{\sigma(2)}+2, K-1, a_\ell, \dots, a_3, a_2+1) - N(K-1, a_\ell, \dots, a_3, a_2+1).\end{aligned}$$

Applying (7.4.1) to the first and second terms in this expression gives

$$\begin{aligned}&KN(k_{\sigma(2)}+2, K-1, a_\ell, \dots, a_3, a_2+1)\\ &= K(k_{\sigma(2)}+2)N(K-1, a_\ell, \dots, a_3, a_2+1) + KN(a_\ell, \dots, a_3, a_2+1),\\ &N(K-1, a_\ell, \dots, a_3, a_2+1) = (K-1)N(a_\ell, \dots, a_3, a_2+1) + N(a_{\ell-1}, \dots, a_3, a_2+1).\end{aligned}$$

Applying (7.4.1) once more to the first term of the first of these identities gives

$$\begin{aligned}&K(k_{\sigma(2)}+2)N(K-1, a_\ell, \dots, a_3, a_2+1)\\ &= K(K-1)(k_{\sigma(2)}+2)N(a_\ell, \dots, a_3, a_2+1) + K(k_{\sigma(2)}+2)N(a_{\ell-1}, \dots, a_3, a_2+1).\end{aligned}$$

Substituting this last identity back into the first of the two identities above, and then substituting the resulting expressions into the first displayed formula for the left-hand side of (7.4.7), the left-hand side becomes

$$\begin{aligned}&\Big(K(K-1)(k_{\sigma(2)}+2)+1\Big)N(a_\ell, \dots, a_3, a_2+1)\\ &\quad + \Big(K(k_{\sigma(2)}+2)-1\Big)N(a_{\ell-1}, \dots, a_3, a_2+1).\end{aligned} \tag{7.4.10}$$

Next apply Lemma 7.4.10 (1) to the first and second terms on the right-hand side of (7.4.7). This gives

$$\begin{aligned}&(K(k_{\sigma(2)}+2) - k_{\sigma(2)} - 1)N(K-1, a_1, \dots, a_\ell)\\ &\quad = K(K(k_{\sigma(2)}+2) - k_{\sigma(2)} - 1)N(a_2+1, \dots, a_\ell)\\ &\qquad - (K(k_{\sigma(2)}+2) - k_{\sigma(2)} - 1)N(a_3, \dots, a_\ell),\\ &(k_{\sigma(2)}+2)N(K-1, a_1, \dots, a_{\ell-1})\\ &\quad = K(k_{\sigma(2)}+2)N(a_2+1, \dots, a_{\ell-1}) - (k_{\sigma(2)}+2)N(a_3, \dots, a_{\ell-1}).\end{aligned}$$

Substituting these two identities into the right-hand side of (7.4.7) gives

$$\begin{aligned}&K\Big(((K-1)(k_{\sigma(2)}+2)+1)N(a_2+1, \dots, a_\ell) + (k_{\sigma(2)}+2)N(a_2+1, \dots, a_{\ell-1})\Big)\\ &\quad - \Big(((K-1)(k_{\sigma(2)}+2)+1)N(a_3, \dots, a_\ell) + (k_{\sigma(2)}+2)N(a_3, \dots, a_{\ell-1})\Big) - k_{\frac{p-1}{p}}.\end{aligned} \tag{7.4.11}$$

It is enough to prove that the difference between (7.4.10) and (7.4.11) is zero. Let this difference be E. Then

$$\begin{aligned}E &= (K-1)N(a_2+1, \dots, a_\ell) + N(a_2+1, \dots, a_{\ell-1})\\ &\quad - ((K-1)(k_{\sigma(2)}+2)+1)N(a_3, \dots, a_\ell) - (k_{\sigma(2)}+2)N(a_3, \dots, a_{\ell-1}) - k_{\frac{p-1}{p}}.\end{aligned} \tag{7.4.12}$$

Applying Corollary 7.4.9 to $t = \frac{p-1}{p}$ gives $a_2 + 1 = 2 + k_{\sigma(2)}$. Therefore applying (7.4.1) to the first and second terms above gives

$$\begin{aligned}(K-1)N(a_2+1, \dots, a_\ell) &= (K-1)(2+k_{\sigma(2)})N(a_3, \dots, a_\ell) + (K-1)N(a_4, \dots, a_\ell),\\ N(a_2+1, \dots, a_{\ell-1}) &= (2+k_{\sigma(2)})N(a_3, \dots, a_{\ell-1}) + N(a_4, \dots, a_{\ell-1}).\end{aligned}$$

Substituting these identities into (7.4.12) gives

$$E = (K-1)N(a_4, \dots, a_\ell) + N(a_4, \dots, a_{\ell-1}) - N(a_3, \dots, a_\ell) - k_{\frac{p-1}{p}}. \tag{7.4.13}$$

Since $a_3 = K - 1$, applying (7.4.1) to the third term gives

$$N(a_3, \dots, a_\ell) = (K-1)N(a_4, \dots, a_\ell) + N(a_5, \dots, a_\ell).$$

Substituting this into (7.4.13), we obtain

$$E = N(a_4, \dots, a_{\ell-1}) - N(a_5, \dots, a_\ell) - k_{\frac{p-1}{p}}.$$

Taking the symmetry in Remark 7.4.3 (5) into account, Lemma 7.4.10 (2) gives $E = 0$. Hence (7.4.7) follows. The identities (7.4.8) and (7.4.9) are proved by the same transformations.

Finally, consider the remaining cases $t \in (0, 1)$. If $s(r) = (K-1, a_1, \dots, a_\ell)$ and $s(s) = (K-1, b_1, \dots, b_m)$, then Corollary 7.4.9 gives

$$s(t) = (K-1, 1, b_m - 1, b_{m-1}, \dots, b_1, K-1, a_\ell, \dots, a_3, a_2 + 1).$$

Put

$$\mu := (b_m - 1, b_{m-1}, \dots, b_1, K-1, a_\ell, \dots, a_3).$$

Then

$$F_{s(t)} = \begin{bmatrix} N(K-1, 1, \mu, a_2+1) & N(K-1, 1, \mu) \\ N(1, \mu, a_2+1) & N(1, \mu) \end{bmatrix}.$$

On the other hand, since $C_t = C_r C_s - D_t$, we compare the entries of $F_{s(t)}$ with those of C_t after applying the induction hypothesis to C_r and C_s. For the same reason as above, it is enough to compare the $(1,1)$, $(2,1)$, and $(1,2)$ entries. Thus it remains to prove the following three identities:

$$\begin{aligned}(7.4.14)\quad N(K-1, 1, \mu, a_2+1) &= N(K-1, a_1, \dots, a_\ell)N(K-1, b_1, \dots, b_m) \\ &\quad + N(K-1, a_1, \dots, a_{\ell-1})N(b_1, \dots, b_m) - k_t,\end{aligned}$$

$$(7.4.15)\quad N(1, \mu, a_2+1) = N(a_1, \dots, a_\ell)N(K-1, b_1, \dots, b_m) + N(a_1, \dots, a_{\ell-1})N(b_1, \dots, b_m),$$

$$(7.4.16)\quad N(1, \mu) = N(a_1, \dots, a_\ell)N(b_1, \dots, b_m) + N(a_1, \dots, a_{\ell-1})N(b_1, \dots, b_{m-1}) - k_t.$$

We prove (7.4.14). Since $b_1 = 1$, applying Lemma 7.4.11 to the left-hand side gives

$$\begin{aligned}(7.4.17)\quad N(K-1, 1, \mu, a_2+1) &= N(K-1, 1, b_m - 1, b_{m-1}, \dots, b_2+1)N(K, a_\ell, \dots, a_2+1) \\ &\quad - N(K-1, 1, b_m-1, b_{m-1}, \dots, b_3)N(a_\ell, \dots, a_2+1).\end{aligned}$$

Applying (7.4.1) to the terms containing K and Lemma 7.4.11 to the terms containing $K - 1$, we rewrite (7.4.17) as

$$\begin{aligned}(7.4.18)\quad & K^2 N(a_2+1, \dots, a_\ell)N(b_2+1, \dots, b_m) + KN(a_2+1, \dots, a_{\ell-1})N(b_2+1, \dots, b_m) \\ & - KN(a_2+1, \dots, a_\ell)N(b_2+1, \dots, b_{m-1}) - N(a_2+1, \dots, a_{\ell-1})N(b_2+1, \dots, b_{m-1}) \\ & - KN(a_2+1, \dots, a_\ell)N(b_3, \dots, b_m) + N(a_2+1, \dots, a_\ell)N(b_3, \dots, b_{m-1}).\end{aligned}$$

On the other hand, applying Lemma 7.4.11 to the right-hand side of (7.4.14) gives

$$\begin{aligned}(7.4.19)\quad & K^2 N(a_2+1, \dots, a_\ell)N(b_2+1, \dots, b_m) - KN(a_2+1, \dots, a_\ell)N(b_3, \dots, b_m) \\ & - KN(a_3, \dots, a_\ell)N(b_2+1, \dots, b_m) + N(a_3, \dots, a_\ell)N(b_3, \dots, b_m) \\ & + KN(a_2+1, \dots, a_{\ell-1})N(b_1, \dots, b_m) - N(a_3, \dots, a_{\ell-1})N(b_2+1, \dots, b_m) - k_t.\end{aligned}$$

It is enough to prove that the difference between (7.4.18) and (7.4.19) is zero. Let this difference be E. Then

(7.4.20)

$$\begin{aligned}E = & - KN(a_2+1, \dots, a_\ell)N(b_2+1, \dots, b_{m-1}) - N(a_2+1, \dots, a_{\ell-1})N(b_2+1, \dots, b_{m-1}) \\ & + N(a_2+1, \dots, a_\ell)N(b_3, \dots, b_{m-1}) + KN(a_3, \dots, a_\ell)N(b_2+1, \dots, b_m) \\ & - N(a_3, \dots, a_\ell)N(b_3, \dots, b_m) + N(a_3, \dots, a_{\ell-1})N(b_2+1, \dots, b_m) + k_t.\end{aligned}$$

Here Lemma 7.4.10 (1) gives

$$N(K-1, b_1, \dots, b_j) = K\,N(b_2+1, \dots, b_j) - N(b_3, \dots, b_j).$$

Applying this for $j = m - 1$ and $j = m$, we can rewrite (7.4.20) as

(7.4.21)
$$\begin{aligned} E = & -N(a_2+1,\dots,a_\ell)N(K-1,b_1,\dots,b_{m-1}) - N(a_2+1,\dots,a_{\ell-1})N(b_1,\dots,b_{m-1}) \\ & + N(a_3,\dots,a_\ell)N(K-1,b_1,\dots,b_m) + N(a_3,\dots,a_{\ell-1})N(b_1,\dots,b_m) + k_t. \end{aligned}$$

Thus (7.4.20) has been rewritten in the form (7.4.21). Furthermore, Lemma 7.4.12 gives

$$\begin{aligned} & N(a_3,\dots,a_\ell)N(K-1,b_1,\dots,b_m) + N(a_3,\dots,a_{\ell-1})N(b_1,\dots,b_m) \\ & \qquad = N(a_3,\dots,a_\ell,K-1,b_1,\dots,b_m), \\ & N(a_2+1,\dots,a_\ell)N(K-1,b_1,\dots,b_{m-1}) + N(a_2+1,\dots,a_{\ell-1})N(b_1,\dots,b_{m-1}) \\ & \qquad = N(a_2+1,\dots,a_\ell,K-1,b_1,\dots,b_{m-1}). \end{aligned}$$

Therefore

$$E = N(a_3,\dots,a_\ell,K-1,b_1,\dots,b_m) - N(a_2+1,\dots,a_\ell,K-1,b_1,\dots,b_{m-1}) + k_t. \tag{7.4.22}$$

Reversing the sequences in (7.4.22) and using Proposition 6.1.9 (1), we can rewrite E as

$$E = N(b_m,\dots,b_1,K-1,a_\ell,\dots,a_3) - N(b_{m-1},\dots,b_1,K-1,a_\ell,\dots,a_3,a_2+1) + k_t. \tag{7.4.23}$$

Finally, Lemma 7.4.10 (2) gives

$$N(b_m,\dots,b_1,K-1,a_\ell,\dots,a_3) = N(b_{m-1},\dots,b_1,K-1,a_\ell,\dots,a_3,a_2+1) - k_t.$$

Hence $E = 0$, and (7.4.14) follows. The identities (7.4.15) and (7.4.16) are proved by the same transformations.

We next prove the case $t \in (1,\infty]$. The case $C_{\frac{1}{0}}$ is checked by direct computation. Hence assume $t \in (1,\infty)$. By Corollary 5.3.7 and Proposition 5.4.4, we have $m^*_{\frac{1}{t}} = m_t$ and $u^*_{\frac{1}{t}} = m_t - u_t - k_t$. Combining this with Theorem 7.2.1, we obtain

$$C_t = \begin{bmatrix} (K-1)m^*_{\frac{1}{t}} + u^*_{\frac{1}{t}} & * \\ m^*_{\frac{1}{t}} & m^*_{\frac{1}{t}} - u^*_{\frac{1}{t}} - k_t \end{bmatrix}$$

(the $(1,2)$ entry is determined by $\det C_t = 1$). Since $\frac{1}{t} \in (0,1)$, write $s^*(\frac{1}{t}) = (b_0,\dots,b_m)$. The result already proved for the interval $(0,1)$ gives

$$C_t = \begin{bmatrix} (K-1)N(b_1,\dots,b_m) + N(b_1,\dots,b_{m-1}) & * \\ N(b_1,\dots,b_m) & N(b_1,\dots,b_m) - N(b_1,\dots,b_{m-1}) - k_t \end{bmatrix}.$$

On the other hand, by Remark 7.4.3 (7), we may write $s(t) = (b_0, b_m,\dots,b_1)$, and hence

$$F_{s(t)} = \begin{bmatrix} N(b_0,b_m,\dots,b_1) & * \\ N(b_m,\dots,b_1) & N(b_m,\dots,b_2) \end{bmatrix}.$$

If $C^*_{\frac{1}{t}}$ denotes the (k_1,k_2,k_3,σ^*)-GC matrix attached to $\frac{1}{t}$, then

$$\operatorname{tr}(C_t) = \operatorname{tr}(C^*_{\frac{1}{t}}) = Km_t - k_t.$$

Also, using $b_0 = K-1$ and $b_1 = 1$, Lemma 7.4.10 (1), and (7.4.1), we obtain

$$\begin{aligned} & \operatorname{tr}(F_{s^*(\frac{1}{t})}) \\ & = N(b_0,b_1,\dots,b_m) + N(b_1,\dots,b_{m-1}) \\ & = KN(b_2+1,\dots,b_m) - N(b_3,\dots,b_m) + N(b_1,\dots,b_{m-1}) \\ & = (K-1)N(b_2+1,\dots,b_m) + N(b_2+1,\dots,b_m) - N(b_3,\dots,b_m) + N(b_1,\dots,b_{m-1}) \\ & = N(b_0,b_m,\dots,b_1) + N(b_2+1,\dots,b_m) - N(b_3,\dots,b_m) \\ & = N(b_0,b_m,\dots,b_1) + N(1,b_2,\dots,b_m) - N(b_3,\dots,b_m) \\ & = N(b_0,b_m,\dots,b_1) + N(b_2,\dots,b_m) \\ & = \operatorname{tr}(F_{s(t)}). \end{aligned}$$

Together with the first part, which gives $\mathrm{tr}(F_{s^*(\frac{1}{t})}) = \mathrm{tr}(C^*_{\frac{1}{t}})$, this also gives $\mathrm{tr}(F_{s(t)}) = \mathrm{tr}(C_t)$. Therefore, to prove $C_t = F_{s(t)}$, it remains to check the equality of the $(1,1)$ and $(2,1)$ entries. The equality of the $(2,1)$ entries follows from Proposition 6.1.9 (1). The equality of the $(1,1)$ entries follows by applying Proposition 6.1.9 (1) after reversing the sequence in the expression for C_t, and then using (7.4.1). This proves the claim. □

REMARK 7.4.13. The matrices $C_{\frac{0}{1}}$ and $F_{s(\frac{0}{1})}$ do not coincide. Indeed, $C_{\frac{0}{1}}$ has a negative entry.

CHAPTER 8

Generalized Discrete Markov Spectra

By Chapter 7 we have prepared several tools: generalized Markov numbers, characteristic numbers, GM distances, generalized Cohn matrices, and generalized strongly admissible sequences. In this chapter these tools are related to the Lagrange and Markov spectra. More precisely, we define a discrete family of values naturally obtained from generalized Markov numbers and show that these values are realized as both Lagrange constants and Markov constants. Thus the arithmetic, combinatorial, geometric, and matrix-theoretic data constructed in Part II are brought together in the language of spectra.

We first define the generalized discrete Markov spectrum as a set of values determined by generalized Markov numbers and then state the main result discussed in this chapter. We then show that each element of this spectrum is realized both as the Lagrange constant of a quadratic irrational and as the Markov constant of a binary quadratic form with rational coefficients. This gives explicit values inside the Lagrange and Markov spectra. We also specialize the general theory to the case $(k_1, k_2, k_3) = (0, 0, 0)$ and recover the classical Markov theorem. After that, we pass from rational slopes to irrational slopes and prove that the bi-infinite sequences obtained from irrational-slope lines give the boundary value $3+k_1+k_2+k_3$. Finally, we discuss the relation between the $(0, 0, 0)$-type and the $(2, 2, 2)$-type, and then consider a natural generalization of Frobenius's uniqueness conjecture.

The material in this chapter is based primarily on [**Gyo25**]. The proof of Markov's theorem follows the basic strategy of the traditional accounts [**Bom07, Aig13, Reu19**], but we present it in terms of right and left mechanical words in order to make the relation with strongly admissible sequences explicit.

1. Definitions and Main Theorems

For $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$, define

$$\mathcal{M}_{k_1,k_2,k_3,\sigma} := \left\{ \frac{\sqrt{((3+k_1+k_2+k_3)m_t - k_t)^2 - 4}}{m_t} \;\middle|\; \begin{array}{l} t \in [0,\infty] \cap \mathbb{Q}, \\ m_t \text{ is a } (k_1,k_2,k_3,\sigma)\text{-GM number} \end{array} \right\},$$

$$\mathcal{M}_{k_1,k_2,k_3} := \bigcup_{\sigma \in \mathfrak{S}_3} \mathcal{M}_{k_1,k_2,k_3,\sigma}.$$

Here m_t is understood together with its component position i_t, and k_t means k_{i_t}. We call $\mathcal{M}_{k_1,k_2,k_3}$ the (k_1, k_2, k_3)-*generalized discrete Markov spectrum*. The central result of this section is the following.

Theorem 8.1.1. *Fix $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$. We use the conventions $1/0 = \infty$ and $1/\infty = 0$. For each reduced fraction $t \in [0, \infty] \cap \mathbb{Q}$, let (m_t, i_t) denote the corresponding (k_1, k_2, k_3, σ)-GM number together with its component position, put $k_t := k_{i_t}$, and let $s(t)$ be the corresponding generalized strongly admissible sequence. For a finite sequence S of positive integers, write $\alpha_S = [\overline{S}]$. Then*

$$\mathcal{L}(\alpha_{s(t)}) = \mathcal{L}(\alpha_{s^*(\frac{1}{t})}) = \frac{\sqrt{((3+k_1+k_2+k_3)m_t - k_t)^2 - 4}}{m_t}.$$

In particular, $\mathcal{M}_{k_1,k_2,k_3} \subset \mathcal{L}$.

The next theorem is an immediate consequence of Theorem 4.4.2.

THEOREM 8.1.2. *With the notation of Theorem 8.1.1, for a finite sequence S of positive integers put $Q_S = (x - \alpha_S y)(x - \alpha'_S y)$, where α'_S is the quadratic conjugate of α_S. Then, for every reduced fraction $t \in [0,\infty] \cap \mathbb{Q}$,*

$$\mathcal{M}(Q_{s(t)}) = \mathcal{M}(Q_{s^*(\frac{1}{t})}) = \frac{\sqrt{((3+k_1+k_2+k_3)m_t - k_t)^2 - 4}}{m_t}.$$

Theorem 8.1.1 follows from the material developed so far as follows.

PROOF OF THEOREM 8.1.1. It suffices to show that the periodic parts of $\overline{s(t)}$ and $\overline{s^*(1/t)}$ agree up to reversal and cyclic shift. Indeed, by Theorem 3.3.3 and the invariance of $\mathcal{S}$ under reversal, this implies $\mathcal{L}(\alpha_{s(t)}) = \mathcal{L}(\alpha_{s^*(1/t)})$. The required agreement follows immediately from Remark 7.4.3 (7). We next prove the formula for $\mathcal{L}(\alpha_{s(t)})$. When $t = 0$, the assertion follows by directly computing $s(0)$, $F_{s(0)}$, and m_0 from the definitions. Hence we assume $t \in (0,\infty]$. By Theorem 3.3.5,

$$\mathcal{L}(\alpha_{s(t)}) = \max\left\{ \frac{\sqrt{(\mathrm{tr}(F_{S_i}))^2 - (-1)^{n+1}\cdot 4}}{(F_{S_i})_{21}} \;\middle|\; 0 \le i \le n \right\}$$

where $s(t) = (a_0,\dots,a_n)$ and $S_i = (a_i,\dots,a_n,a_0,\dots,a_{i-1})$. By Remark 7.4.3 (0), n is odd, and hence $(-1)^{n+1} = 1$. For $i = 0$ we have $F_{S_0} = F_{s(t)}$, which equals C_t by Theorem 7.4.4. Moreover, Theorem 7.2.1 gives $\mathrm{tr}(C_t) = (3+k_1+k_2+k_3)m_t - k_t$ and $(F_{S_0})_{21} = (C_t)_{21} = m_t$. Thus it remains to prove that $\min\{(F_{S_i})_{21} \mid 0 \le i \le n\} = (F_{S_0})_{21}$. This is the only remaining point to check. Since $(F_{S_0})_{21} = N(a_1,\dots,a_n)$, it suffices to show that this number is the minimum of

$$N_t = \{N(w) \mid \text{there exists } 0 \le k \le n \text{ such that } w = (a_{k+1},a_{k+2},\dots,a_{k+n})\}.$$

Project $\widetilde{\mathbb{R}^2}$ to the triangulation of the once-punctured torus. Then $\overline{L_t}$ corresponds to a loop on the torus; see also Remark 7.4.1. When $\overline{L_t}$ is regarded as such a loop and $w = (a_{k+1},\dots,a_{k+n})$, we denote by $\overline{L_t}(w)$ the part of $\overline{L_t}$ whose sign sequence is w, as determined by the triangle-crossing and edge-crossing rules.

If the endpoints of $\overline{L_t}(w)$ are modified as shown in Table 1, the triangle-crossing, edge-crossing, and endpoint rules give an arc $\widetilde{L_t}(w)$ whose sign sequence is w.

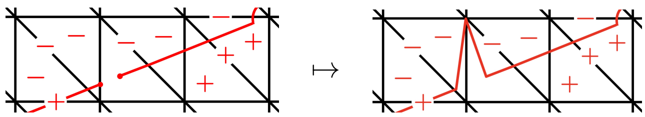


FIGURE 1. Example of the modification from $\overline{L_t}(w)$ to $\widetilde{L_t}(w)$

The endpoint modification connects the two endpoints of $\overline{L_t}(w)$ without introducing any unnecessary detour within the homotopy class on the once-punctured torus; see Figure 1. This can be checked by considering all endpoint-modification patterns.

Therefore, if $t = p/q$, then $\widetilde{L_t}(w)$ can be represented as a curve segment from $(0,0)$ to (q,p). The resulting curves $\widetilde{L_t}(w)$ are generalized arcs. In particular, $\widetilde{L_t}(a_1,\dots,a_n)$ is the segment L_t with endpoints $A = (0,0)$ and $B = (q,p)$, hence corresponds to γ^L_{AB}. By Theorem 6.3.7, this implies that $N(a_1,\dots,a_n)$ is the minimum of N_t, and the proof is complete. $\square$

The key point of the proof is the following. After reducing the possible values of the Lagrange constant to

$$\mathcal{L}(\alpha_{s(t)}) = \max\left\{ \frac{\sqrt{(\mathrm{tr}(F_{S_i}))^2 - (-1)^{n+1}\cdot 4}}{(F_{S_i})_{21}} \;\middle|\; 0 \le i \le n \right\}$$

one has to determine which cyclic shift S_i gives the smallest value of $(F_{S_i})_{21}$, and what that value is. In [**Bom07**], this step is handled by ad hoc inequalities. In this text we instead use the minimality of the GM distance and the fact that the minimal GM distance is a GM number. The former method is tailored to the case of ordinary Markov numbers, whereas the present method treats general GM numbers uniformly.

EXAMPLE 8.1.3. Let $(k_1, k_2, k_3, \sigma) = (1, 2, 0, \mathrm{id})$ and $t = \frac{2}{5}$. Since $i_{2/5} = 1$, we have $s\left(\frac{2}{5}\right) = (5, 1, 3, 3, 1, 5, 4, 1, 3, 4)$. Therefore the sequences w giving the elements $N(w) \in N_{2/5}$ are

$$(1,3,3,1,5,4,1,3,4),\ (3,3,1,5,4,1,3,4,5),\ (3,1,5,4,1,3,4,5,1),\ (1,5,4,1,3,4,5,1,3),$$
$$(5,4,1,3,4,5,1,3,3),\ (4,1,3,4,5,1,3,3,1),\ (1,3,4,5,1,3,3,1,5),\ (3,4,5,1,3,3,1,5,4),$$
$$(4,5,1,3,3,1,5,4,1),\ (5,1,3,3,1,5,4,1,3)$$

respectively. The corresponding values of $N(w)$ are

$$8227,\ 32957,\ 12039,\ 12041,\ 32937,\ 8261,\ 9997,\ 31881,\ 12199,\ 11127,$$

and the minimum of $N_{2/5}$ is 8227. The corresponding arcs $\overline{L_t}(w)$ and $\widetilde{L_t}(w)$ are listed in Table 2, which is collected at the end of this section.

$$F_{s(\frac{2}{5})} = \begin{bmatrix} N(5,1,3,3,1,5,4,1,3,4) & N(5,1,3,3,1,5,4,1,3) \\ N(1,3,3,1,5,4,1,3,4) & N(1,3,3,1,5,4,1,3) \end{bmatrix} = \begin{bmatrix} 47431 & 11127 \\ 8227 & 1930 \end{bmatrix}$$

Thus $\alpha_{s(2/5)} = (\sqrt{2436508317} + 45501)/16454$ and

$$Q_{s(\frac{2}{5})} = x^2 - \frac{45501}{8227}xy - \frac{11127}{8227}y^2.$$

For these values we have

$$\mathcal{L}\left(\frac{\sqrt{2436508317} + 45501}{16454}\right) = \mathcal{M}\left(x^2 - \frac{45501}{8227}xy - \frac{11127}{8227}y^2\right) = \frac{\sqrt{2436508317}}{8227}.$$

EXAMPLE 8.1.4. We now list several quadratic irrationals α and the corresponding values $\mathcal{L}(\alpha)$ obtained from Theorem 8.1.1. Table 3 gives the case $(k_1, k_2, k_3) = (0, 0, 0)$; when $k_1 = k_2 = k_3$, the value is independent of $\sigma \in \mathfrak{S}_3$. Tables 4, 5, and 6 give the case $(0, 0, 1)$. Tables 7, 8, and 9 give the case $(0, 1, 1)$. Table 10 gives the case $(1, 1, 1)$, Table 11 gives the case $(2, 2, 2)$, and Table 12 gives the case $(k_1, k_2, k_3, \sigma) = (1, 2, 0, \mathrm{id})$. In each list, the entries are ordered by the corresponding GM number m_t.

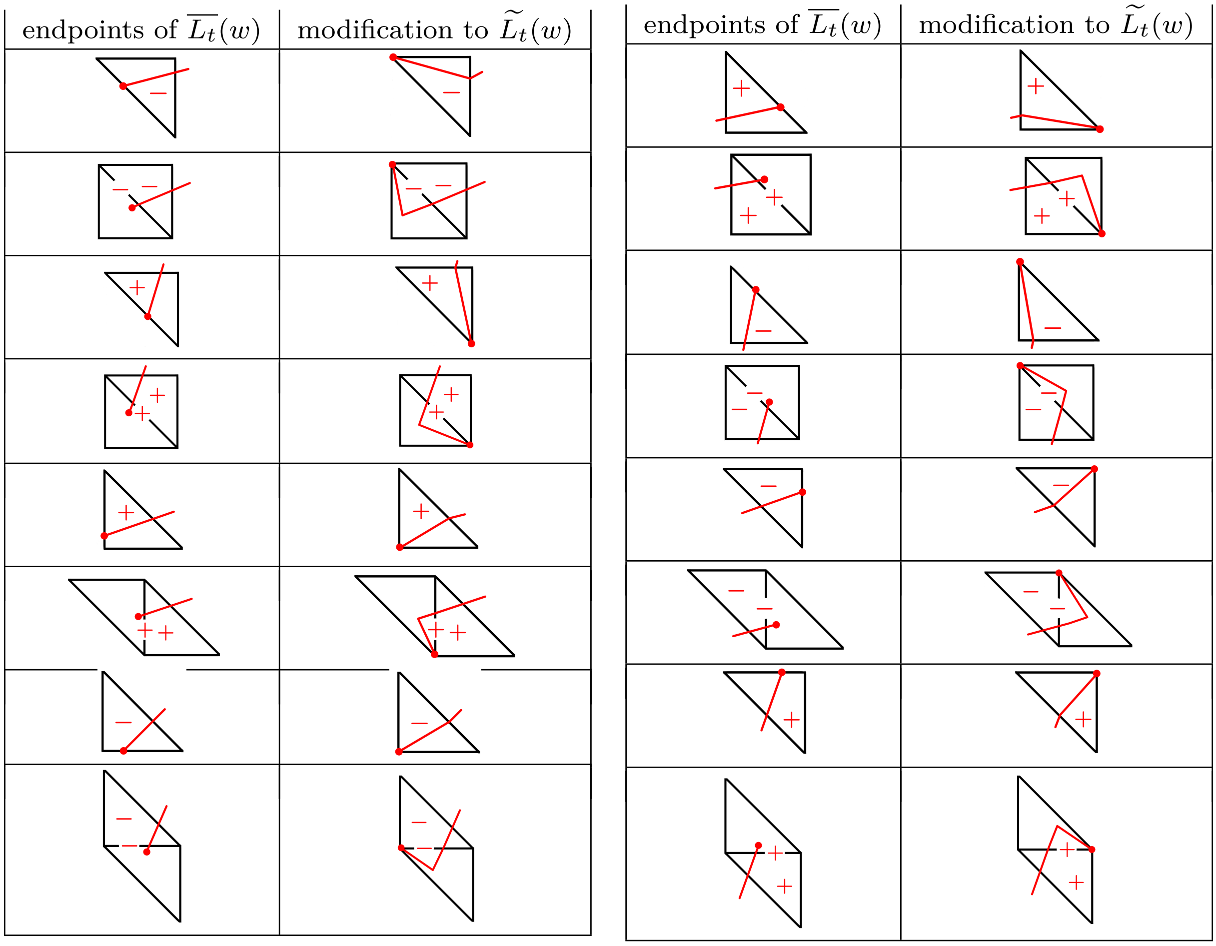


TABLE 1. Endpoint modification

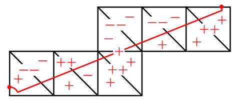
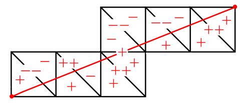
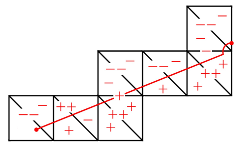
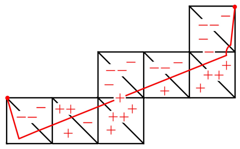
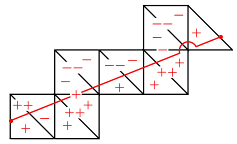
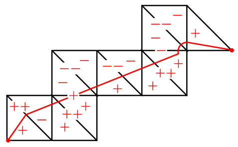
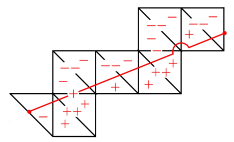
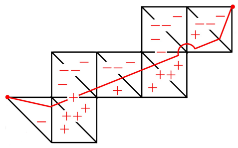
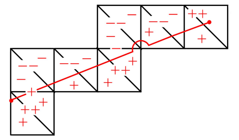
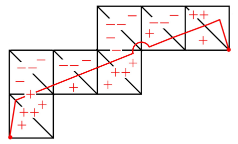
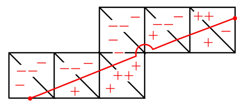
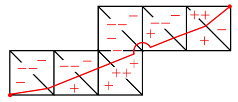
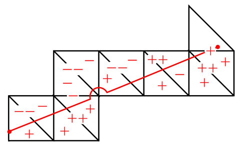
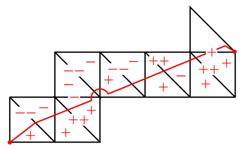
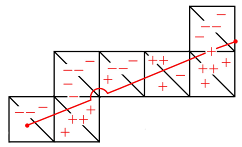
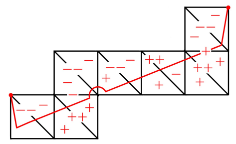
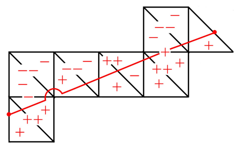
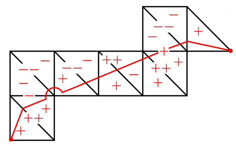
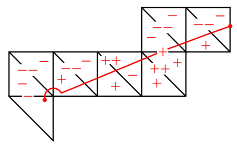
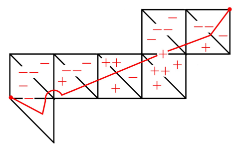
w	$\overline{L_t}(w)$	$\widetilde{L_t}(w)$	$N(w)$
$(1,3,3,1,5,4,1,3,4)$			8227
$(3,3,1,5,4,1,3,4,5)$			32957
$(3,1,5,4,1,3,4,5,1)$			12039
$(1,5,4,1,3,4,5,1,3)$			12041
$(5,4,1,3,4,5,1,3,3)$			32937
$(4,1,3,4,5,1,3,3,1)$			8261
$(1,3,4,5,1,3,3,1,5)$			9997
$(3,4,5,1,3,3,1,5,4)$			31881
$(4,5,1,3,3,1,5,4,1)$			12199
$(5,1,3,3,1,5,4,1,3)$			11127

TABLE 2. $\overline{L_t}(w)$ and $\widetilde{L_t}(w)$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(1,1)$	$\frac{\sqrt{5}+1}{2}$	1	$\sqrt{5}$
$\frac{1}{1}$	$(2,2)$	$\sqrt{2}+1$	2	$2\sqrt{2}$
$\frac{1}{2}$	$(2,1,1,2)$	$\frac{\sqrt{221}+11}{10}$	5	$\frac{\sqrt{221}}{5}$
$\frac{1}{3}$	$(2,1,1,1,1,2)$	$\frac{\sqrt{1517}+29}{26}$	13	$\frac{\sqrt{1517}}{13}$
$\frac{2}{3}$	$(2,1,1,2,2,2)$	$\frac{\sqrt{7565}+63}{58}$	29	$\frac{\sqrt{7565}}{29}$
$\frac{1}{4}$	$(2,1,1,1,1,1,1,2)$	$\frac{5\sqrt{26}+19}{17}$	34	$\frac{10\sqrt{26}}{17}$
$\frac{1}{5}$	$(2,1,1,1,1,1,1,1,1,2)$	$\frac{\sqrt{71285}+199}{178}$	89	$\frac{\sqrt{71285}}{89}$
$\frac{3}{4}$	$(2,1,1,2,2,2,2,2)$	$\frac{\sqrt{257045}+367}{338}$	169	$\frac{\sqrt{257045}}{169}$

TABLE 3. $(k_1, k_2, k_3) = (0, 0, 0)$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(2,1)$	$\sqrt{3}+1$	1	$2\sqrt{3}$
$\frac{1}{1}$	$(3,2)$	$\frac{\sqrt{15}+3}{2}$	2	$\sqrt{15}$
$\frac{1}{2}$	$(3,1,1,3)$	$\frac{5\sqrt{29}+23}{14}$	7	$\frac{5\sqrt{29}}{7}$
$\frac{1}{3}$	$(3,1,2,1,1,3)$	$\frac{7\sqrt{51}+43}{25}$	25	$\frac{14\sqrt{51}}{25}$
$\frac{2}{3}$	$(3,1,1,3,3,2)$	$\frac{\sqrt{11235}+83}{53}$	53	$\frac{2\sqrt{11235}}{53}$
$\frac{1}{4}$	$(3,1,2,1,1,2,1,3)$	$\frac{\sqrt{15293}+107}{62}$	93	$\frac{\sqrt{15293}}{31}$
$\frac{1}{5}$	$(3,1,2,1,2,1,1,2,1,3)$	$\frac{3\sqrt{53207}+599}{346}$	346	$\frac{3\sqrt{53207}}{173}$
$\frac{3}{4}$	$(3,1,1,3,2,3,3,2)$	$\frac{\sqrt{308765}+435}{278}$	417	$\frac{2\sqrt{308765}}{278}$

TABLE 4. $(k_1, k_2, k_3, \sigma) = (0, 0, 1, \mathrm{id})$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(2,1)$	$\sqrt{3}+1$	1	$2\sqrt{3}$
$\frac{1}{1}$	$(3,3)$	$\frac{\sqrt{13}+3}{2}$	3	$\sqrt{13}$
$\frac{1}{2}$	$(3,1,2,3)$	$\frac{\sqrt{399}+17}{10}$	10	$\frac{\sqrt{399}}{5}$
$\frac{1}{3}$	$(3,1,2,2,1,3)$	$\frac{\sqrt{21605}+127}{74}$	37	$\frac{\sqrt{21605}}{37}$
$\frac{2}{3}$	$(3,1,2,3,3,3)$	$\frac{\sqrt{47523}+185}{109}$	109	$\frac{2\sqrt{47523}}{109}$
$\frac{1}{4}$	$(3,1,2,1,2,2,1,3)$	$\frac{5\sqrt{3003}+237}{137}$	137	$\frac{10\sqrt{3003}}{137}$
$\frac{1}{5}$	$(3,1,2,1,2,2,1,2,1,3)$	$\frac{\sqrt{4173845}+1769}{1022}$	511	$\frac{\sqrt{4173845}}{511}$
$\frac{3}{4}$	$(3,1,2,3,3,3,3,3)$	$\frac{\sqrt{5654883}+2018}{1189}$	1189	$\frac{2\sqrt{5654883}}{1189}$

TABLE 5. $(k_1, k_2, k_3, \sigma) = (0, 0, 1, (1\ 2\ 3))$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(1,1)$	$\frac{\sqrt{5}+1}{2}$	1	$\sqrt{5}$
$\frac{1}{1}$	$(3,2)$	$\frac{\sqrt{15}+3}{2}$	2	$\sqrt{15}$
$\frac{1}{2}$	$(3,1,1,2)$	$\frac{3\sqrt{11}+8}{5}$	5	$\frac{6\sqrt{11}}{5}$
$\frac{1}{3}$	$(3,1,1,1,1,2)$	$\frac{15\sqrt{3}+21}{13}$	13	$\frac{30\sqrt{3}}{13}$
$\frac{1}{4}$	$(3,1,1,1,1,1,1,2)$	$\frac{\sqrt{4623}+55}{34}$	34	$\frac{\sqrt{4623}}{17}$
$\frac{2}{3}$	$(3,1,1,3,2,2)$	$\frac{\sqrt{2669}+41}{26}$	39	$\frac{\sqrt{2669}}{13}$
$\frac{1}{5}$	$(3,1,1,1,1,1,1,1,1,2)$	$\frac{\sqrt{31683}+144}{89}$	89	$\frac{2\sqrt{31683}}{89}$
$\frac{1}{6}$	$(3,1,1,1,1,1,1,1,1,1,1,2)$	$\frac{\sqrt{217155}+377}{233}$	233	$\frac{2\sqrt{217155}}{233}$

TABLE 6. $(k_1, k_2, k_3, \sigma) = (0, 0, 1, (1\ 3\ 2))$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(3, 1)$	$\frac{\sqrt{21}+3}{2}$	1	$\sqrt{21}$
$\frac{1}{1}$	$(4, 3)$	$\frac{4\sqrt{3}+6}{3}$	3	$\frac{8\sqrt{3}}{3}$
$\frac{1}{2}$	$(4, 1, 2, 4)$	$\frac{\sqrt{1023}+29}{13}$	13	$\frac{2\sqrt{1023}}{13}$
$\frac{1}{3}$	$(4, 1, 3, 2, 1, 4)$	$\frac{3\sqrt{2567}+139}{61}$	61	$\frac{6\sqrt{2567}}{61}$
$\frac{2}{3}$	$(4, 1, 2, 4, 4, 3)$	$\frac{\sqrt{49506}+195}{89}$	178	$\frac{2\sqrt{49506}}{89}$
$\frac{1}{4}$	$(4, 1, 3, 1, 2, 3, 1, 4)$	$\frac{44\sqrt{273}+666}{291}$	291	$\frac{88\sqrt{273}}{291}$
$\frac{1}{5}$	$(4, 1, 3, 1, 3, 2, 1, 3, 1, 4)$	$\frac{531\sqrt{43}+3191}{1393}$	1393	$\frac{1062\sqrt{43}}{1393}$
$\frac{3}{4}$	$(4, 1, 2, 4, 3, 4, 4, 3)$	$\frac{2\sqrt{9600702}+5431}{2479}$	2479	$\frac{4\sqrt{9600702}}{2479}$

TABLE 7. $(k_1, k_2, k_3, \sigma) = (0, 1, 1, \mathrm{id})$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(2, 1)$	$\sqrt{3}+1$	1	$2\sqrt{3}$
$\frac{1}{1}$	$(4, 3)$	$\frac{4\sqrt{3}+6}{3}$	3	$\frac{8\sqrt{3}}{3}$
$\frac{1}{2}$	$(4, 1, 2, 3)$	$\frac{2\sqrt{39}+11}{5}$	10	$\frac{4\sqrt{39}}{5}$
$\frac{1}{3}$	$(4, 1, 2, 2, 1, 3)$	$\frac{\sqrt{8463}+82}{37}$	37	$\frac{2\sqrt{8463}}{37}$
$\frac{1}{4}$	$(4, 1, 2, 1, 2, 2, 1, 3)$	$\frac{\sqrt{469221}+611}{274}$	137	$\frac{\sqrt{469221}}{137}$
$\frac{2}{3}$	$(4, 1, 2, 4, 3, 3)$	$\frac{2\sqrt{30102}+305}{139}$	139	$\frac{4\sqrt{30102}}{139}$
$\frac{1}{5}$	$(4, 1, 2, 1, 2, 2, 1, 2, 1, 3)$	$\frac{6\sqrt{45298}+1140}{511}$	511	$\frac{12\sqrt{45298}}{511}$
$\frac{2}{5}$	$(4, 1, 2, 2, 1, 4, 3, 1, 2, 3)$	$\frac{22\sqrt{43662}+4050}{1839}$	1839	$\frac{44\sqrt{43662}}{1839}$

TABLE 8. $(k_1, k_2, k_3, \sigma) = (0, 1, 1, (1\ 2\ 3))$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(2,1)$	$\sqrt{3}+1$	1	$2\sqrt{3}$
$\frac{1}{1}$	$(4,2)$	$\sqrt{6}+2$	2	$2\sqrt{6}$
$\frac{1}{2}$	$(4,1,1,3)$	$\frac{12\sqrt{2}+15}{7}$	7	$\frac{24\sqrt{2}}{7}$
$\frac{1}{3}$	$(4,1,2,1,1,3)$	$\frac{\sqrt{15621}+111}{50}$	25	$\frac{\sqrt{15621}}{25}$
$\frac{2}{3}$	$(4,1,1,4,3,2)$	$\frac{4\sqrt{1743}+138}{67}$	67	$\frac{8\sqrt{1743}}{67}$
$\frac{1}{4}$	$(4,1,2,1,1,2,1,3)$	$\frac{\sqrt{53823}+207}{93}$	93	$\frac{2\sqrt{53823}}{93}$
$\frac{1}{5}$	$(4,1,2,1,2,1,1,2,1,3)$	$\frac{12\sqrt{1299}+386}{173}$	346	$\frac{24\sqrt{1299}}{173}$
$\frac{3}{4}$	$(4,1,1,4,2,3,4,2)$	$\frac{\sqrt{2729103}+1356}{661}$	661	$\frac{2\sqrt{2729103}}{661}$

TABLE 9. $(k_1, k_2, k_3, \sigma) = (0, 1, 1, (1\ 3\ 2))$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(3,1)$	$\frac{\sqrt{21}+3}{2}$	1	$\sqrt{21}$
$\frac{1}{1}$	$(5,3)$	$\frac{\sqrt{285}+15}{6}$	3	$\frac{\sqrt{285}}{3}$
$\frac{1}{2}$	$(5,1,2,4)$	$\frac{5\sqrt{237}+71}{26}$	13	$\frac{5\sqrt{237}}{13}$
$\frac{1}{3}$	$(5,1,3,2,1,4)$	$\frac{11\sqrt{1101}+339}{122}$	61	$\frac{11\sqrt{1101}}{61}$
$\frac{2}{3}$	$(5,1,2,5,4,3)$	$\frac{\sqrt{1692597}+1167}{434}$	217	$\frac{\sqrt{1692597}}{217}$
$\frac{1}{4}$	$(5,1,3,1,2,3,1,4)$	$\frac{\sqrt{3045021}+1623}{582}$	291	$\frac{\sqrt{3045021}}{291}$
$\frac{1}{5}$	$(5,1,3,1,3,2,1,3,1,4)$	$\frac{\sqrt{69839445}+7775}{2786}$	1393	$\frac{\sqrt{69839445}}{1393}$
$\frac{3}{4}$	$(5,1,2,5,3,4,5,3)$	$\frac{\sqrt{485629365}+19735}{7346}$	3673	$\frac{\sqrt{485629365}}{3673}$

TABLE 10. $(k_1, k_2, k_3) = (1, 1, 1)$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(5,1)$	$\frac{3\sqrt{5}+5}{2}$	1	$3\sqrt{5}$
$\frac{1}{1}$	$(8,4)$	$3\sqrt{2}+4$	4	$6\sqrt{2}$
$\frac{1}{2}$	$(8,1,3,6)$	$\frac{3\sqrt{221}+43}{10}$	25	$\frac{3\sqrt{221}}{5}$
$\frac{1}{3}$	$(8,1,5,3,1,6)$	$\frac{3\sqrt{1517}+113}{26}$	169	$\frac{3\sqrt{1517}}{13}$
$\frac{2}{3}$	$(8,1,3,8,6,4)$	$\frac{3\sqrt{7565}+247}{58}$	841	$\frac{3\sqrt{7565}}{29}$
$\frac{1}{4}$	$(8,1,5,1,3,5,1,6)$	$\frac{15\sqrt{26}+74}{17}$	1156	$\frac{30\sqrt{26}}{17}$
$\frac{1}{5}$	$(8,1,5,1,5,3,1,5,1,6)$	$\frac{3\sqrt{71285}+775}{178}$	7921	$\frac{3\sqrt{71285}}{89}$
$\frac{3}{4}$	$(8,1,3,8,4,6,8,4)$	$\frac{3\sqrt{257045}+1439}{338}$	28561	$\frac{3\sqrt{257045}}{169}$

TABLE 11. $(k_1, k_2, k_3) = (2, 2, 2)$

t	$s(t)$	$\alpha = [\overline{s(t)}]$	m_t	$\mathcal{L}(\alpha)$
$\frac{0}{1}$	$(3,1)$	$\frac{\sqrt{21}+3}{2}$	1	$\sqrt{21}$
$\frac{1}{1}$	$(5,4)$	$\frac{\sqrt{30}+5}{2}$	4	$\sqrt{30}$
$\frac{1}{2}$	$(5,1,3,4)$	$\frac{10\sqrt{26}+47}{17}$	17	$\frac{20\sqrt{26}}{17}$
$\frac{1}{3}$	$(5,1,3,3,1,4)$	$\frac{\sqrt{723}+25}{9}$	81	$\frac{2\sqrt{723}}{9}$
$\frac{2}{3}$	$(5,1,3,5,4,4)$	$\frac{\sqrt{5004165}+2061}{746}$	373	$\frac{\sqrt{5004165}}{373}$
$\frac{1}{4}$	$(5,1,3,1,3,3,1,4)$	$\frac{\sqrt{1340963}+1077}{386}$	386	$\frac{\sqrt{1340963}}{193}$
$\frac{1}{5}$	$(5,1,3,1,3,3,1,3,1,4)$	$\frac{\sqrt{16635}+120}{43}$	1849	$\frac{2\sqrt{16635}}{43}$
$\frac{3}{4}$	$(5,1,3,5,4,4,5,4)$	$\frac{2\sqrt{150737006}+22602}{8185}$	8185	$\frac{4\sqrt{150737006}}{8185}$

TABLE 12. $(k_1, k_2, k_3, \sigma) = (1, 2, 0, \mathrm{id})$

2. Mechanical Words

In this section we introduce the mechanical words needed for the proof of Markov's theorem in the next section. Since our goal is to relate them to strongly admissible sequences, we define them using slopes at least 1, rather than the more usual convention in which the slope is at most 1.

DEFINITION 8.2.1. Let $t \in [1, \infty]$ and $\theta \in \mathbb{R}$. First suppose that $1 \le t < \infty$. Orient the line $\ell_{t,\theta} : y = tx + \theta$ in the direction in which both coordinates increase, that is, from lower left to upper right. For each $n \in \mathbb{Z}$, let $P_n = ((n-\theta)/t, n)$ be the intersection of $\ell_{t,\theta}$ with the horizontal line $y = n$. Let

$$r_n = \left\lceil \frac{n-\theta}{t} \right\rceil$$

be the x-coordinate of the nearest lattice point on or to the right of P_n, and let

$$l_n = \left\lfloor \frac{n-\theta}{t} \right\rfloor$$

be the x-coordinate of the nearest lattice point on or to the left of P_n. If P_n itself is a lattice point, that lattice point is regarded as belonging to both the right and the left side. Put

$$\varepsilon_n^R = r_{n+1} - r_n, \qquad \varepsilon_n^L = l_{n+1} - l_n.$$

Since $t \ge 1$, we have $\varepsilon_n^R, \varepsilon_n^L \in \{0, 1\}$.

The bi-infinite word $\mathbf{b}^R = (b_n^R)_{n \in \mathbb{Z}}$ defined by

$$b_n^R = \begin{cases} X & \text{if } \varepsilon_n^R = 1, \\ Y & \text{if } \varepsilon_n^R = 0 \end{cases}$$

is called the *right mechanical word* of slope t and intercept θ. Similarly, the bi-infinite word $\mathbf{b}^L = (b_n^L)_{n \in \mathbb{Z}}$ defined by

$$b_n^L = \begin{cases} X & \text{if } \varepsilon_n^L = 1, \\ Y & \text{if } \varepsilon_n^L = 0 \end{cases}$$

is called the *left mechanical word* of slope t and intercept θ. Right and left mechanical words are collectively called *mechanical words*.

Finally, for $t = \infty$, we define $\cdots YYY \cdots$ to be the mechanical word of slope ∞.

EXAMPLE 8.2.2. For the line $\ell_{\frac{5}{2},\frac{1}{4}} : y = \frac{5}{2}x + \frac{1}{4}$, the right mechanical word is the bi-infinite purely periodic word with period $XYXYY$. The left mechanical word is the same word; see the left panel of Figure 2. In this example the left and right mechanical words coincide, but the situation changes when the line passes through a lattice point. For $\ell_{\frac{5}{2},0} : y = \frac{5}{2}x$, the right mechanical word again has period $XYXYY$, while the left mechanical word has period $YYXYX$; see the middle panel of Figure 2. These two words agree after a shift, but the example shows where the distinction between the two conventions comes from. If the slope is irrational and the line passes through a lattice point, then the left and right mechanical words differ only around the unique lattice point through which the line passes; see the right panel of Figure 2.

We record several elementary properties of mechanical words.

LEMMA 8.2.3. *Let $t = \frac{p}{q} \in [1, \infty)$ be a reduced fraction. Then every mechanical word of slope t has period p, and one period contains exactly q occurrences of X. Thus, if w is one period, then*

$$\frac{|w|}{|w|_X} = \frac{p}{q} = t.$$

PROOF. It is enough to prove the assertion for right mechanical words; the proof for left mechanical words is identical. We have

$$r_n = \left\lceil \frac{n-\theta}{t} \right\rceil = \left\lceil \frac{q(n-\theta)}{p} \right\rceil.$$

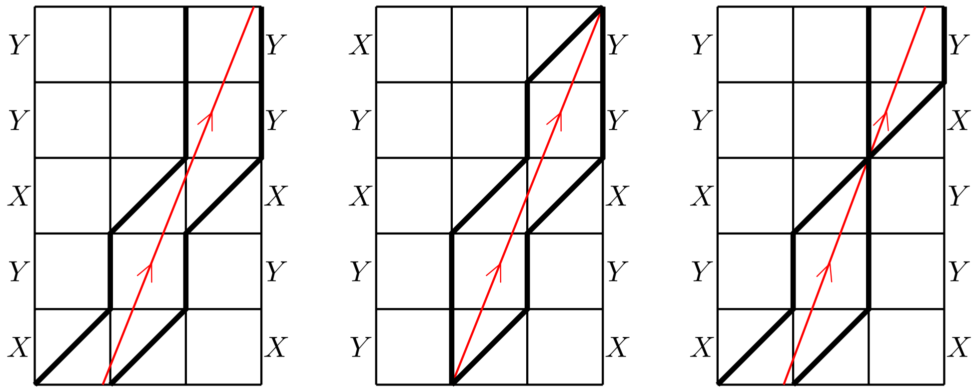


FIGURE 2. Pieces of mechanical words, read from bottom to top

Since $t = \frac{p}{q}$,

$$r_{n+p} = \left\lceil \frac{q(n+p-\theta)}{p} \right\rceil = \left\lceil \frac{q(n-\theta)}{p} + q \right\rceil = r_n + q.$$

Therefore

$$\varepsilon^R_{n+p} = r_{n+p+1} - r_{n+p} = r_{n+1} - r_n = \varepsilon^R_n,$$

so p is a period. Moreover, the number of X's in one period is

$$\sum_{n=0}^{p-1} \varepsilon^R_n = r_p - r_0 = q.$$

Hence $|w|/|w|_X = p/q = t$. The same argument, with the floor function in place of the ceiling function, proves the assertion for left mechanical words. □

Next we relate shifts of mechanical words to changes of intercept.

LEMMA 8.2.4. *Let $k \in \mathbb{Z}$, and define the shift of a word by*

$$(T^k\mathbf{b})_n = b_{n+k}.$$

Then, for every $t \in [1, \infty]$ and every intercept θ,

$$T^k\mathbf{b}^R(t,\theta) = \mathbf{b}^R(t, \theta - k), \qquad T^k\mathbf{b}^L(t,\theta) = \mathbf{b}^L(t, \theta - k).$$

In particular, a shift of a mechanical word is again a mechanical word of the same slope.

PROOF. The case $t = \infty$ is clear because the word is $\cdots YYY \cdots$. Assume $1 \le t < \infty$. For right mechanical words, put

$$r_n(\theta) = \left\lceil \frac{n-\theta}{t} \right\rceil.$$

Then

$$r_n(\theta - k) = \left\lceil \frac{n-(\theta-k)}{t} \right\rceil = \left\lceil \frac{n+k-\theta}{t} \right\rceil = r_{n+k}(\theta).$$

Therefore

$$r_{n+1}(\theta - k) - r_n(\theta - k) = r_{n+k+1}(\theta) - r_{n+k}(\theta),$$

and hence $T^k\mathbf{b}^R(t,\theta) = \mathbf{b}^R(t, \theta - k)$. The proof for left mechanical words is identical, using $l_n(\theta) = \lfloor (n-\theta)/t \rfloor$. □

The converse is not true in general: mechanical words with different intercepts need not be related by a shift, because the proof above accounts only for integer shifts of the intercept. For rational slopes, however, the converse holds. We first show that, for rational slopes, left mechanical words may be rewritten as right mechanical words.

LEMMA 8.2.5. *Assume that $t \in [1, \infty]$ is rational. Then every mechanical word of slope t can be written as a right mechanical word of the same slope t.*

PROOF. There is nothing to prove if the word is already a right mechanical word. Let $\mathbf{b} = \mathbf{b}^L(t,\theta)$ be a left mechanical word of slope t. If $t = \infty$, then $\mathbf{b} = \cdots YYY \cdots$, which is also a right mechanical word. Assume $1 \le t < \infty$, and write $t = p/q$ in lowest terms, with $p, q \in \mathbb{Z}_{>0}$ and $p \ge q$. The left mechanical word is determined by the differences of

$$l_n = \left\lfloor \frac{n-\theta}{t} \right\rfloor = \left\lfloor \frac{q(n-\theta)}{p} \right\rfloor.$$

Put $x_n = (n-\theta)/t$. Since $t = p/q$, we have $x_{n+p} = x_n + q$. Hence the fractional parts $\{x_n\}$ are periodic in n with period p, and the set

$$\{\{x_n\} \mid n \in \mathbb{Z}\}$$

is finite. Choose $\delta > 0$ sufficiently small so that

$$\{x_n\} + \delta < 1$$

for all n with $\{x_n\} \ne 0$. Such a δ exists because the set of fractional parts is finite; for example one may take

$$0 < \delta < \min\{\, 1 - \{x_n\} \mid n = 0, \ldots, p-1,\ \{x_n\} \ne 0 \,\},$$

with arbitrary $0 < \delta < 1$ if the set on the right is empty. Then, for all $n \in \mathbb{Z}$,

$$\lceil x_n + \delta \rceil = \lfloor x_n \rfloor + 1.$$

Indeed, this is immediate if x_n is an integer, and otherwise follows from $\{x_n\} + \delta < 1$. Put $\theta' = \theta - t\delta$. Then

$$\frac{n-\theta'}{t} = \frac{n-\theta}{t} + \delta = x_n + \delta.$$

For the sequence

$$r_n = \left\lceil \frac{n-\theta'}{t} \right\rceil$$

defining the right mechanical word $\mathbf{b}^R(t,\theta')$, we have $r_n = l_n + 1$ for every n. Taking differences gives $r_{n+1} - r_n = l_{n+1} - l_n$. Thus $\mathbf{b}^L(t,\theta)$ and $\mathbf{b}^R(t,\theta')$ give the same letter at every position, i.e. $\mathbf{b}^L(t,\theta) = \mathbf{b}^R(t,\theta')$. □

LEMMA 8.2.6. *Let $t \in [1,\infty]$ be rational. Then the mechanical word of slope t is uniquely determined up to shift.*

PROOF. For $t = \infty$ the only word is $\cdots YYY \cdots$, so the assertion is clear. Write $t = p/q$ in lowest terms, with $p, q \in \mathbb{Z}_{>0}$ and $p \ge q$. By Lemma 8.2.5, it suffices to consider right mechanical words. The right mechanical word $\mathbf{b}^R(t,\theta)$ is determined by the differences of

$$r_n = \left\lceil \frac{n-\theta}{t} \right\rceil.$$

Put $\alpha = q/p$ and $\beta = -q\theta/p$. Then $r_n = \lceil n\alpha + \beta \rceil$, and

$$b_n = X \iff \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil = 1.$$

The right-hand side is unchanged when β is increased by 1, so we regard β as a point of $\mathbb{R}/\mathbb{Z}$. Put

$$u_\beta(n) = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil.$$

Then

$$u_{\beta+\alpha}(n) = \lceil (n+2)\alpha + \beta \rceil - \lceil (n+1)\alpha + \beta \rceil = u_\beta(n+1).$$

Thus increasing β by α corresponds to shifting the word by one position.

Since $\alpha = q/p$ and $\gcd(p,q) = 1$, the points

$$0, \alpha, 2\alpha, \ldots, (p-1)\alpha$$

in $\mathbb{R}/\mathbb{Z}$ are just a permutation of

$$0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}.$$

A right mechanical word is constant on each half-open interval determined by these points, and the map $\beta \mapsto \beta + \alpha$ cyclically permutes these half-open intervals. Therefore all words obtained by changing the intercept agree up to shift. □

PROPOSITION 8.2.7. *Let $t, t' \in [1, \infty] \cap \mathbb{Q}$ with $t \neq t'$. Let $\mathbf{b}$ be a mechanical word of slope t, and let $\mathbf{b}'$ be a mechanical word of slope t'. Then $\mathbf{b}$ and $\mathbf{b}'$ are not shift-equivalent, and they are not reversals of each other up to shift.*

PROOF. For a periodic word $\mathbf{c}$, let $D_X(\mathbf{c})$ denote the proportion of X's in one period. This is independent of the choice of period, and is unchanged by shifts and reversal.

The mechanical word of slope $t = \infty$ is $\cdots YYY \cdots$, so $D_X(\mathbf{b}) = 0$ in this case. If $t = p/q \in [1, \infty) \cap \mathbb{Q}$ is written in lowest terms, Lemma 8.2.3 gives $D_X(\mathbf{b}) = q/p = 1/t$. Thus, in general, $D_X(\mathbf{b}) = 1/t$, with the convention $1/\infty = 0$.

If $\mathbf{b}$ and $\mathbf{b}'$ were shift-equivalent, then $D_X(\mathbf{b}) = D_X(\mathbf{b}')$, and hence $1/t = 1/t'$, so $t = t'$, contradicting the assumption. Thus they are not shift-equivalent.

Similarly, if $\mathbf{b}$ and $\mathbf{b}'$ were reversals of each other up to shift, that is, if $\mathbf{b} = T^k((\mathbf{b}')^*)$ for some $k \in \mathbb{Z}$, then reversal and shift would again preserve D_X. Hence $D_X(\mathbf{b}) = D_X(\mathbf{b}')$, forcing $t = t'$, again a contradiction. □

Finally, in this section, we relate rational-slope mechanical words to strongly admissible sequences in the case $(k_1, k_2, k_3) = (0, 0, 0)$. This relation is the key point in the proof of Markov's theorem.

PROPOSITION 8.2.8. *Let $\mathbf{w}$ be a mechanical word of rational slope t, allowing $t = 1/0$. Replace X by $2, 2$ and Y by $1, 1$, and denote the resulting sequence by $\widehat{\mathbf{w}}$. Then, in the case $(k_1, k_2, k_3) = (0, 0, 0)$, one has*

$$\widehat{\mathbf{w}} = (\ldots, s(t), s(t), s(t), \ldots),$$

where $s(t)$ is the corresponding generalized strongly admissible sequence.

PROOF. The case $t = 1/0$ follows by direct inspection. Assume $1 \leq t < \infty$. By Lemma 8.2.5, any rational-slope mechanical word may be written as a right mechanical word of the same slope, so we may assume that $\mathbf{w}$ is right mechanical. Since $\mathbf{w}$ has rational slope t, we may take as its defining line the line obtained by extending $\overline{L_t}$ infinitely upward and downward. Applying the rule for right mechanical words to $\overline{L_t}$, we see that on an interval contributing the letter X, the line $\overline{L_t}$ passes through four triangles of $\widetilde{\mathbb{R}^2}$ in the x-direction. The sign rule assigns the signs $-, -, +, +$ to these triangles. On an interval contributing the letter Y, the line passes through two triangles in the y-direction, and the sign rule assigns the signs $-, +$; see Figure 3. Therefore, along the period of $\mathbf{w}$ corresponding to $\overline{L_t}$, assigning $2, 2$ to X and $1, 1$ to Y gives precisely the sign sequence determined by the triangle sign rule. By the definition of $s(t)$, this proves the claim. □

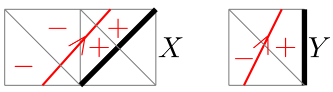


FIGURE 3. The relation between the sign rule and right mechanical words

3. Markov's Theorem

In this section we consider the case $(k_1, k_2, k_3) = (0, 0, 0)$. In this case the $(1, 1)$-, $(1, 2)$-, and $(1, 3)$-entries at each vertex of $\mathrm{M}\mathbb{T}(0, 0, 0, \sigma)$ do not depend on $\sigma \in \mathfrak{S}_3$, so the fraction labeling of GM numbers is the same for every choice of σ. We therefore omit σ. Applying Theorem 8.1.1 in this case gives $\mathcal{M}_{0,0,0} \subset \mathcal{L} \cap (0, 3) \subset \mathcal{M} \cap (0, 3)$. The reverse inclusion is the classical theorem known as Markov's theorem, which we now prove.

THEOREM 8.3.1 (Markov's Theorem). $\mathcal{M}_{0,0,0} = \mathcal{L} \cap (0, 3) = \mathcal{M} \cap (0, 3)$.

It suffices to prove $\mathcal{M}_{0,0,0} \supset \mathcal{M} \cap (0,3)$. By Corollary 4.3.4, an element of $\mathcal{M} \cap (0,3)$ is represented by a bi-infinite sequence $\mathbf{b}$ of positive integers with $\mathcal{S}(\mathbf{b}) \in (0,3)$. Thus we have to identify the possible forms of such sequences. Since $\mathcal{S}(\mathbf{b}) > 0$ for every such sequence, we study the condition $\mathcal{S}(\mathbf{b}) \leq 3$. From the definition

$$\mathcal{S}(\mathbf{b}) := \sup_{h \in \mathbb{Z}} \ell_h(\mathbf{b}) = \sup_{h \in \mathbb{Z}} ([b_h; b_{h+1}, \dots] + [0; b_{h-1}, b_{h-2}, \dots]).$$

the following observation is immediate.

Proposition 8.3.2. *If $\mathcal{S}(\mathbf{b}) \leq 3$, then $b_i = 1$ or $b_i = 2$ for every $i \in \mathbb{Z}$.*

In the remainder of the proof, we work under the assumption that each b_i is either 1 or 2. We introduce the notation

$$\ell(\dots, b_{h-2}, b_{h-1} \mid b_h, b_{h+1}, \dots) := [b_h; b_{h+1}, \dots] + [0; b_{h-1}, b_{h-2}, \dots].$$

This lets us write computations of $\ell_h(\mathbf{b})$ without specifying the index h explicitly.

Lemma 8.3.3. *For one-sided infinite sequences u, v and $c \in \mathbb{Z}_{\geq 1}$, $\ell(u^* \mid c, v) = \ell(v^* \mid c, u)$. Here u^* denotes the reversal of u. In particular, $\mathcal{S}(\mathbf{b}) \leq 3$ if and only if $\mathcal{S}(\mathbf{b}^*) \leq 3$.*

Proof. This follows immediately from the computation

$$\ell(u^* \mid c, v) = [c; v] + [0; u] = \left(c + \frac{1}{[u]}\right) + \frac{1}{[v]} = \left(c + \frac{1}{[v]}\right) + \frac{1}{[u]} = [c; u] + [0; v] = \ell(v^* \mid c, u).$$

□

We also have the following characterization.

Proposition 8.3.4. *The condition $\mathcal{S}(\mathbf{b}) \leq 3$ is equivalent to the following two conditions on $\mathbf{b}$.*

(i) *The sequence $\mathbf{b}$ contains neither the subsequence $(1,2,1)$ nor the subsequence $(2,1,2)$.*
(ii) *If $\mathbf{b}$ or $\mathbf{b}^*$ can be written as $(u^*, 1, 1, 2, 2, v)$ using one-sided infinite sequences u, v, then $[v] \leq [u]$.*

Proof. We first show that $\mathcal{S}(\mathbf{b}) \leq 3$ implies (i) and (ii). Computing $\ell(u^*, 1 \mid 2, 1, v)$ and using $u, v \geq 1$, we obtain

$$\ell(u^*, 1 \mid 2, 1, v) = [2; 1, v] + [0; 1, u] > [2; 1, 1] + [0; 1, 1] = \frac{5}{2} + \frac{1}{2} = 3.$$

Thus $(1,2,1)$ cannot occur in $\mathbf{b}$. The displayed inequality follows from (2.2.5); although u and v need not be integers, the same inequality remains valid when the final component is a positive real number. If $\mathbf{b}$ contains $(2,1,2)$, the preceding result lets us extend it to $(2,2,1,2)$. Then

$$\ell(u^*, 2 \mid 2, 1, 2, v) = [2; 1, 2, v] + [0; 2, u] > [2; 1, 2] + [0; 2, 1] = \frac{8}{3} + \frac{1}{3} = 3.$$

This proves (i). If $[v] > [u]$, then

$$\ell(u^*, 1, 1 \mid 2, 2, v) = [2; 2, v] + [0; 1, 1, u] > [2; 2, v] + [0; 1, 1, v] = \frac{5[v] + 2}{2[v] + 1} + \frac{[v] + 1}{2[v] + 1} = 3.$$

so (ii) also holds.

Conversely, assume (i) and (ii). We distinguish the three possible cuts: (1) $\ell(u^* \mid 1, v)$, (2) $\ell(u^*, 1 \mid 2, v)$, and (3) $\ell(u^*, 2 \mid 2, v)$. In case (1), condition (ii) gives

$$\ell(u^* \mid 1, v) = [1; v] + [0; u] < [1; 1] + [0; 1] = 3.$$

Hence the required inequality holds.

In case (2), since neither $(1,2,1)$ nor $(2,1,2)$ occurs, we may write $u = (1, u')$ and $v = (2, v')$. Hence

$$\ell(u^*, 1 \mid 2, v) = \ell((u')^*, 1, 1 \mid 2, 2, v') = [2; 2, v'] + [0; 1, 1, u'] \leq [2; 2, u'] + [0; 1, 1, u'] = 3.$$

Thus the inequality holds.

In case (3), we further divide into (3-1) $v = (1, v')$ and (3-2) $v = (2, v')$. In case (3-1), the absence of $(2,1,2)$ implies $v' = (1, v'')$. By Lemma 8.3.3,

$$\ell(u^*, 2 \mid 2, 1, v') = \ell(u^*, 2 \mid 2, 1, 1, v'') = \ell((v'')^*, 1, 1 \mid 2, 2, u) \leq 3.$$

where the final inequality is obtained by applying the same argument as in case (2) to $\mathbf{b}^*$. In case (3-2),

$$\ell(u^*, 2 \mid 2, 2, v') < [2; 2, v'] + [0; 2, u] < [2; 2] + [0; 2] = 3.$$

and the proof is complete. □

REMARK 8.3.5. Condition (ii) will be used below in the following form. If $x, y \in \{1, 2\}$ and $\mathbf{b} = (\dots, x, w^*, 1, 1, 2, 2, w, y, \dots)$, then

$$x = 1, y = 2 \Rightarrow |w| \text{ is odd}, \quad x = 2, y = 1 \Rightarrow |w| \text{ is even}$$

This is just a reformulation of condition (ii).

When we write 1^n or 2^m as part of a sequence, it means that 1 appears n times consecutively or that 2 appears m times consecutively.

THEOREM 8.3.6. *If $\mathcal{S}(\mathbf{b}) \leq 3$, then* $\mathbf{b}$ *is one of the following types.*

(1) $(\overline{1}^*, 2, 2, \overline{1})$ *or* $(\overline{2}^*, 1, 1, \overline{2})$ *(degenerate type)*
(2) $(\overline{1}^*, \overline{1})$ *or* $(\overline{2}^*, \overline{2})$ *(constant type)*
(3) $(\dots, 1^{m_{i-1}}, 2^{n_{i-1}}, 1^{m_i}, 2^{n_i}, 1^{m_{i+1}}, 2^{n_{i+1}}, \dots)$ *(regular type)*

where m_j and n_j are even for every j.

PROOF. First we show that whenever a finite block of consecutive 1's occurs in $\mathbf{b}$, its length is even. Suppose that $\mathbf{b} = (\dots, 2, 1^m, 2^n, 1, \dots)$ and that m is odd. If $n = 1$, then $(1,2,1)$ occurs, contradicting Proposition 8.3.4 (i). Assume $n \geq 3$, allowing also $n = \infty$. If $m = 1$, then $(2,1,2)$ occurs, again a contradiction. If $m \geq 3$, then $(1,1,1,2,2,2)$ occurs; considering $\ell(\dots, 1, 1, 1 \mid 2, 2, 2, \dots)$ contradicts Proposition 8.3.4 (ii). Hence $n = 2$. In this case

$$\mathbf{b} = (\dots, 2, 1^{m-2}, 1, 1, 2, 2, 1^p, 2^{n'}, \dots)$$

and Proposition 8.3.4 (ii) implies that $p \leq m - 2$ and that p must be odd. Repeating the preceding argument gives $n' = 2$. Iterating this argument would force the lengths of consecutive blocks of 1's to decrease strictly as one moves to the right, which is impossible in a bi-infinite sequence. Hence m is even. By applying the same argument to the reversed sequence and using Lemma 8.3.3, one sees that every finite block of consecutive 2's also has even length. This proves that the numbers m_j, n_j in the regular type are even.

Now suppose that $\mathbf{b}$ has infinitely many 1's to the left but is not of constant type; thus $\mathbf{b} = (\overline{1}^*, 2, 2, v)$. If $v = (\overline{1})$, this is degenerate type, so assume otherwise.

Write $\mathbf{b} = (\overline{1}^*, 2, 2, 1^p, 2, \dots)$, allowing $p = 0$. By Proposition 8.3.4 (ii), p must be odd, contradicting what was proved above. Therefore the only non-constant sequence with infinitely many 1's to the left is the degenerate one. The case of infinitely many 2's to the left is analogous.

This proves the theorem. □

We next compute the values for the degenerate and constant types.

PROPOSITION 8.3.7. *If* $\mathbf{b}$ *is of degenerate type, then* $\mathcal{S}(\overline{1}^*, 2, 2, \overline{1}) = \mathcal{S}(\overline{2}^*, 1, 1, \overline{2}) = 3$. *If* $\mathbf{b}$ *is of constant type, then* $\mathcal{S}(\overline{1}^*, \overline{1}) = \sqrt{5}$ *and* $\mathcal{S}(\overline{2}^*, \overline{2}) = 2\sqrt{2}$.

PROOF. The values $\mathcal{S}(\overline{1}^*, \overline{1})$ and $\mathcal{S}(\overline{2}^*, \overline{2})$ are obtained directly from the definition, since every cut gives the same decomposition.

We show that $\mathcal{S}(\overline{1}^*, 2, 2, \overline{1}) = 3$.

$$\ell(\overline{1}^*, 1 \mid 2, 2, \overline{1}) = [2; 2, \overline{1}] + [0; 1, 1, \overline{1}] = \frac{5[\overline{1}] + 2}{2[\overline{1}] + 1} + \frac{[\overline{1}] + 1}{2[\overline{1}] + 1} = 3$$

By this computation and Lemma 8.3.3,

$$\ell(\overline{1}^*, 1, 2 \mid 2, \overline{1}) = 3$$

Moreover,

$$\ell(u^*, 1 \mid 1, v) = [1; v] + [0; 1, u] < 2 + 1 = 3$$

Thus only cuts between 1 and 2 need be considered. The remaining pattern is $\ell(\overline{1}^*, 2, 2 \mid 1, \overline{1})$, which equals $\ell(\overline{1}^*, 1 \mid 1, 2, 2, \overline{1})$ by Lemma 8.3.3 and is less than 3.

Next we prove $\mathcal{S}(\overline{2}^*, 1, 1, \overline{2}) = 3$.

$$\ell(\overline{2}^*, 1, 1 \mid 2, \overline{2}) = [2; 2, \overline{2}] + [0; 1, 1, \overline{2}] = \frac{5[\overline{2}] + 2}{2[\overline{2}] + 1} + \frac{[\overline{2}] + 1}{2[\overline{2}] + 1} = 3$$

By this computation and Lemma 8.3.3,

$$\ell(\overline{2}^*, 2 \mid 2, 1, 1, \overline{2}) = 3$$

Moreover,

$$\ell(u^*, 2 \mid 2, 2, v) = [2; 2, v] + [0; 2, u] < \frac{5}{2} + \frac{1}{2} = 3$$

Together with the preceding equality, this shows that every cut between two 2's gives a value at most 3. The inequality $\ell(\overline{2}^*, 1 \mid 1, \overline{2}) < 3$ was proved in the first part. The remaining pattern is $\ell(\overline{2}^*, 2 \mid 1, 1, \overline{2})$, which equals $\ell(\overline{2}^*, 1 \mid 1, \overline{2})$ by Lemma 8.3.3 and is therefore less than 3. □

By Theorem 8.3.6, if $\mathcal{S}(\mathbf{b}) \leq 3$, then $\mathbf{b}$ can be written using the blocks $(1, 1)$ and $(2, 2)$. We replace $(2, 2)$ by X and $(1, 1)$ by Y. Then the preceding classification becomes

(1) $\overline{X}^* Y \overline{X}$ or $\overline{Y}^* X \overline{Y}$ (degenerate type)
(2) $\overline{X}^* \overline{X}$ or $\overline{Y}^* \overline{Y}$ (constant type)
(3) $\cdots X^{k_{i-1}} Y^{\ell_{i-1}} X^{k_i} Y^{\ell_i} X^{k_{i+1}} Y^{\ell_{i+1}} \cdots$ (regular type)

Proposition 8.3.4 can now be restated as follows.

Proposition 8.3.8. *For a bi-infinite sequence* $\mathbf{b}$, *the following are equivalent.*

(1) $\mathcal{S}(\mathbf{b}) \leq 3$.
(2) *The sequence* $\mathbf{b}$ *can be written as a sequence in the letters* X, Y, *and the following two conditions hold.*
   (i) *If* $\mathbf{b}$ *has a representation* $\mathbf{b} = u^* Y X v$, *then either* $u = v$, *or there exist words* w, u', v' *in* X, Y *such that* $\mathbf{b} = (u')^* X w^* Y X w Y v'$.
   (ii) *If* $\mathbf{b}$ *has a representation* $\mathbf{b} = u^* X Y v$, *then either* $u = v$, *or there exist words* w, u', v' *in* X, Y *such that* $\mathbf{b} = (u')^* Y w^* X Y w X v'$.

Proof. Assume (1). Then $\mathbf{b}$ can be written in the letters X, Y by Theorem 8.3.6. Condition (i) follows from condition (ii) of Proposition 8.3.4 for $\mathbf{b}$, and condition (ii) follows from the same condition for $\mathbf{b}^*$. Conversely, if (2) holds, then condition (i) of Proposition 8.3.4 is automatic because $\mathbf{b}$ is written in X, Y, and its condition (ii) follows from the two displayed conditions above. □

The following consequence will be useful.

Proposition 8.3.9. *Let* $\mathbf{b} = \cdots X^{k_{i-1}} Y^{\ell_{i-1}} X^{k_i} Y^{\ell_i} X^{k_{i+1}} Y^{\ell_{i+1}} \cdots$ *be a regular bi-infinite sequence with* $\mathcal{S}(\mathbf{b}) \leq 3$. *Then either* $k_i = 1$ *for every* $i \in \mathbb{Z}$, *or* $\ell_i = 1$ *for every* $i \in \mathbb{Z}$.

Proof. Suppose that neither alternative holds. Then, for some $m, n \geq 2$,

$$\mathbf{b} = \cdots Y^m (XY)^k X^n \cdots \text{ or } \cdots X^m (YX)^k Y^n \cdots$$

where $k = 0$ is allowed. Choose such a part of $\mathbf{b}$ with k minimal. If $k = 0$, then $\mathbf{b}$ contains either $Y^{m-1} Y X X^{n-1}$ or $X^{m-1} X Y Y^{n-1}$, hence contains $(1, 1, 1, 2, 2, 2)$ or $(2, 2, 2, 1, 1, 1)$, a contradiction. Thus $k \geq 1$. Then

$$\mathbf{b} = \cdots Y^m (XY)^k X^n (YX)^{k'} Y^{m'} \cdots \text{ or } \cdots X^m (YX)^k Y^n (XY)^{k'} X^{m'} \cdots$$

with k' chosen so that $m' \geq 2$. In the first case,

$$\mathbf{b} = \cdots Y^m (XY)^{k-1} X (YX) X^{n-1} (YX)^{k'} Y^{m'} \cdots$$

Applying Proposition 8.3.8 to the central block $X(YX)X^{n-1}$ shows first that $n = 2$; otherwise the block of consecutive X's is too long and violates the required comparison. When $n = 2$, the same proposition applied across the central block implies that the alternating block on the right must have length $k' < k$. This contradicts the minimality of k. The second case is identical after interchanging X and Y. ☐

In the preceding proposition, we call the first case, where $k_i = 1$ for every i, the *X-type*, and the second case, where $\ell_i = 1$ for every i, the *Y-type*.

DEFINITION 8.3.10. Let $\mathbf{b} = \cdots X^{k_{i-1}} Y^{\ell_{i-1}} X^{k_i} Y^{\ell_i} X^{k_{i+1}} Y^{\ell_{i+1}} \cdots$ be a regular bi-infinite sequence; here we do not impose $\mathcal{S}(\mathbf{b}) \leq 3$. Define

$$\begin{aligned} C^X(\mathbf{b}) &:= (\dots, (0)^{k_{i-1}-1}, \ell_{i-1}, (0)^{k_i-1}, \ell_i, (0)^{k_{i+1}-1}, \ell_{i+1}, \dots), \\ C^Y(\mathbf{b}) &:= (\dots, k_{i-1}, (0)^{\ell_{i-1}-1}, k_i, (0)^{\ell_i-1}, k_{i+1}, (0)^{\ell_{i+1}-1}, \dots) \end{aligned}$$

We call these sequences the *X-characteristic sequence* and the *Y-characteristic sequence* of $\mathbf{b}$, respectively.

If $\mathbf{b}$ is of X-type, then the X-characteristic sequence has all entries at least 1, whereas the Y-characteristic sequence consists only of 0's and 1's. For Y-type the roles are reversed. In terms of characteristic sequences, Proposition 8.3.8 can be reformulated as follows; this is a direct check.

PROPOSITION 8.3.11. *For a regular bi-infinite sequence $\mathbf{b}$, the following three conditions are equivalent.*

(1) $\mathcal{S}(\mathbf{b}) \leq 3$.
(2) *The X-characteristic sequence $C^X(\mathbf{b}) = (x_i)_{i\in\mathbb{Z}}$ is defined and, for every $i \in \mathbb{Z}$,*

$$\begin{aligned} (x_i - 1, x_{i+1}, x_{i+2}, \dots) &\preceq (x_{i-1}, x_{i-2}, x_{i-3}, \dots), \\ (x_i - 1, x_{i-1}, x_{i-2}, \dots) &\preceq (x_{i+1}, x_{i+2}, x_{i+3}, \dots) \end{aligned}$$

hold.
(3) *The Y-characteristic sequence $C^Y(\mathbf{b}) = (y_i)_{i\in\mathbb{Z}}$ is defined and, for every $i \in \mathbb{Z}$,*

$$\begin{aligned} (y_i - 1, y_{i+1}, y_{i+2}, \dots) &\preceq (y_{i-1}, y_{i-2}, y_{i-3}, \dots), \\ (y_i - 1, y_{i-1}, y_{i-2}, \dots) &\preceq (y_{i+1}, y_{i+2}, y_{i+3}, \dots) \end{aligned}$$

hold.

Here $\preceq$ denotes lexicographic order.

We now temporarily forget that X and Y stand for $(2,2)$ and $(1,1)$, and consider the free group on the letters X, Y, denoted $\mathfrak{F}(X,Y)$. Define automorphisms $\lambda, \rho \in \operatorname{Aut} \mathfrak{F}(X,Y)$ by their actions on the generators:

$$\lambda\colon \begin{cases} X \mapsto X \\ Y \mapsto XY \end{cases}, \quad \rho\colon \begin{cases} X \mapsto XY \\ Y \mapsto Y \end{cases}$$

Their inverses are

$$\lambda^{-1}\colon \begin{cases} X \mapsto X \\ Y \mapsto X^{-1}Y \end{cases}, \quad \rho^{-1}\colon \begin{cases} X \mapsto XY^{-1} \\ Y \mapsto Y \end{cases}$$

These automorphisms also extend to bi-infinite sequences in the letters X, Y.

LEMMA 8.3.12. *If $\mathcal{S}(\mathbf{b}) \leq 3$, then $\mathcal{S}(\lambda(\mathbf{b})) \leq 3$ and $\mathcal{S}(\rho(\mathbf{b})) \leq 3$. Moreover, if $\mathbf{b}$ is of X-type, then $\mathcal{S}(\rho^{-1}(\mathbf{b})) \leq 3$, and if $\mathbf{b}$ is of Y-type, then $\mathcal{S}(\lambda^{-1}(\mathbf{b})) \leq 3$.*

PROOF. Looking at the characteristic sequences of $\mathbf{b}$, we have

$$C^X(\rho(\mathbf{b})) = C^X(\mathbf{b}) + (\overline{1}^*, \overline{1}), \quad C^Y(\lambda(\mathbf{b})) = C^Y(\mathbf{b}) + (\overline{1}^*, \overline{1})$$

Thus Proposition 8.3.11 gives $\mathcal{S}(\lambda(\mathbf{b})) \leq 3$ and $\mathcal{S}(\rho(\mathbf{b})) \leq 3$. If $\mathbf{b}$ is of X-type (respectively, Y-type), then $C^X(\mathbf{b})$ (respectively, $C^Y(\mathbf{b})$) has all entries at least 1, so $C^X(\rho^{-1}(\mathbf{b}))$ (respectively, $C^Y(\lambda^{-1}(\mathbf{b}))$) is defined and

$$C^X(\rho^{-1}(\mathbf{b})) = C^X(\mathbf{b}) - (\overline{1}^*, \overline{1}), \quad C^Y(\lambda^{-1}(\mathbf{b})) = C^Y(\mathbf{b}) - (\overline{1}^*, \overline{1})$$

Again by Proposition 8.3.11, we obtain $\mathcal{S}(\rho^{-1}(\mathbf{b})) \le 3$ and $\mathcal{S}(\lambda^{-1}(\mathbf{b})) \le 3$, respectively. □

We now use these conditions to describe the inequality $\mathcal{S}(\mathbf{b}) \le 3$ in terms of mechanical words.

LEMMA 8.3.13. *Let* $\mathbf{c}$ *be a mechanical word of slope* $t \in [1,\infty]$. *Then* $\lambda(\mathbf{c})$ *and* $\rho(\mathbf{c})$ *are also mechanical words. More precisely, their slopes are respectively*

$$L(t) = 2 - \frac{1}{t}, \qquad R(t) = t + 1,$$

with the conventions $L(\infty) = 2$ *and* $R(\infty) = \infty$.

PROOF. The case $t = \infty$ is immediate. Indeed, $\mathbf{c} = \cdots YYY \cdots$. Under λ this becomes the alternating word $\cdots XYXYXY \cdots$, which is mechanical of slope 2, while under ρ it remains $\cdots YYY \cdots$, which is mechanical of slope ∞.

The case $t = 1$ is also immediate. Then $\mathbf{c} = \cdots XXX \cdots$. Under λ it remains $\cdots XXX \cdots$, which is mechanical of slope 1, and under ρ it becomes the alternating word $\cdots XYXYXY \cdots$, which is mechanical of slope 2.

Assume $1 < t < \infty$. We first treat the case where $\mathbf{c}$ is a right mechanical word. Write $\mathbf{c} = \mathbf{b}^R(t,\theta)$ and put

$$r_n = \left\lceil \frac{n-\theta}{t} \right\rceil.$$

Then

$$c_n = X \iff r_{n+1} - r_n = 1, \qquad c_n = Y \iff r_{n+1} - r_n = 0.$$

First consider λ. Let P_n be the position at which $\lambda(c_n)$ begins, normalized by $P_0 = 0$. Since $\lambda(X) = X$ and $\lambda(Y) = XY$, we have $P_n = 2n - (r_n - r_0)$. For each n, write

$$r_n = \frac{n-\theta}{t} + \delta_n, \qquad 0 \le \delta_n < 1.$$

Put $t_\lambda = 2 - 1/t$ and $\theta_\lambda = r_0 + \theta/t$. We show that $\lambda(\mathbf{c})$ is the right mechanical word of slope t_λ and intercept θ_λ. Define

$$R_m := \left\lceil \frac{m-\theta_\lambda}{t_\lambda} \right\rceil.$$

Then

$$P_n = 2n - (r_n - r_0) = \left(2 - \frac{1}{t}\right) n + r_0 + \frac{\theta}{t} - \delta_n = t_\lambda n + \theta_\lambda - \delta_n.$$

Thus

$$\frac{P_n - \theta_\lambda}{t_\lambda} = n - \frac{\delta_n}{t_\lambda},$$

and since $0 \le \delta_n/t_\lambda < 1$, we obtain $R_{P_n} = n$.

If $c_n = X$, then $r_{n+1} - r_n = 1$, so $P_{n+1} = P_n + 1$. Hence $R_{P_n+1} - R_{P_n} = R_{P_{n+1}} - R_{P_n} = 1$. This corresponds to $\lambda(c_n) = X$.

If $c_n = Y$, then $r_{n+1} - r_n = 0$, so $P_{n+1} = P_n + 2$. Moreover

$$\frac{P_n + 1 - \theta_\lambda}{t_\lambda} = n + \frac{1-\delta_n}{t_\lambda},$$

and $0 < (1-\delta_n)/t_\lambda < 1$, so $R_{P_n+1} = n+1$. Since we already know $R_{P_k} = k$ for every k, we also have $R_{P_n+2} = R_{P_{n+1}} = n+1$. Therefore

$$R_{P_n+1} - R_{P_n} = 1, \qquad R_{P_n+2} - R_{P_n+1} = 0.$$

This corresponds to $\lambda(c_n) = XY$.

We have shown that

$$\lambda(\mathbf{c}) = \mathbf{b}^R\left(2 - \frac{1}{t},\ r_0 + \frac{\theta}{t}\right).$$

In particular, $\lambda(\mathbf{c})$ is a mechanical word of slope $2 - 1/t$.

Next consider ρ. Let Q_n be the position at which $\rho(c_n)$ begins, normalized by $Q_0 = 0$. Since $\rho(X) = XY$ and $\rho(Y) = Y$, we have $Q_n = n + (r_n - r_0)$. Put $t_\rho = t+1$ and $\theta_\rho = \theta - r_0$. We show that $\rho(\mathbf{c})$ is the right mechanical word of slope t_ρ and intercept θ_ρ. Define

$$S_m := \left\lceil \frac{m-\theta_\rho}{t_\rho} \right\rceil.$$

As above, write $r_n = (n-\theta)/t + \delta_n$. Then

$$\frac{Q_n - \theta_\rho}{t_\rho} = \frac{n + r_n - r_0 - \theta + r_0}{t+1} = r_n - \frac{t\delta_n}{t+1}.$$

Since $0 \le t\delta_n/(t+1) < 1$, we have $S_{Q_n} = r_n$.

If $c_n = Y$, then $r_{n+1} - r_n = 0$, so $Q_{n+1} = Q_n + 1$. Hence $S_{Q_n+1} - S_{Q_n} = S_{Q_{n+1}} - S_{Q_n} = r_{n+1} - r_n = 0$. This corresponds to $\rho(c_n) = Y$.

If $c_n = X$, then $r_{n+1} - r_n = 1$, so $Q_{n+1} = Q_n + 2$. The condition $r_{n+1} - r_n = 1$ is equivalent to $\delta_n < 1/t$. In this case

$$\frac{Q_n + 1 - \theta_\rho}{t_\rho} = r_n + \frac{1 - t\delta_n}{t+1},$$

and $0 < (1 - t\delta_n)/(t+1) < 1$, so $S_{Q_n+1} = r_n + 1$. Moreover $S_{Q_n+2} = S_{Q_{n+1}} = r_{n+1} = r_n + 1$. Therefore

$$S_{Q_n+1} - S_{Q_n} = 1, \qquad S_{Q_n+2} - S_{Q_n+1} = 0.$$

This corresponds to $\rho(c_n) = XY$.

Thus

$$\rho(\mathbf{c}) = \mathbf{b}^R(t+1, \theta - r_0),$$

and in particular $\rho(\mathbf{c})$ is a mechanical word of slope $t+1$.

It remains to consider the case where $\mathbf{c}$ is a left mechanical word. Write $\mathbf{c} = \mathbf{b}^L(t,\theta)$ and put

$$l_n = \left\lfloor \frac{n-\theta}{t} \right\rfloor.$$

The beginning positions of $\lambda(c_n)$ and $\rho(c_n)$ are respectively

$$P_n = 2n - (l_n - l_0), \qquad Q_n = n + (l_n - l_0).$$

Repeating the same calculation with floor functions gives

$$\lambda(\mathbf{c}) = \mathbf{b}^L\left(2 - \frac{1}{t},\, l_0 + \frac{\theta}{t} - \left(1 - \frac{1}{t}\right)\right), \quad \rho(\mathbf{c}) = \mathbf{b}^L(t+1, \theta - l_0 - 1).$$

Thus the assertion also holds for left mechanical words. ☐

Lemma 8.3.14. *Let* $\mathbf{b}$ *be a regular bi-infinite word satisfying* $\mathcal{S}(\mathbf{b}) \le 3$. *Put* $\mathbf{b}^{(0)} = \mathbf{b}$, *and suppose that for every* $N \ge 0$ *there exist* $\sigma_N \in \{\lambda, \rho\}$ *and a bi-infinite word* $\mathbf{b}^{(N+1)}$ *such that*

$$\mathbf{b}^{(N)} = \sigma_N(\mathbf{b}^{(N+1)}).$$

If none of the words $\mathbf{b}^{(N)}$ *is of constant type, then* $\mathcal{S}(\mathbf{b}) = 3$.

Proof. Since $\mathbf{b} = \mathbf{b}^{(0)}$ is regular, it is either of X-type or of Y-type. If it is of X-type, then $\rho^{-1}(\mathbf{b})$ exists; if it is of Y-type, then $\lambda^{-1}(\mathbf{b})$ exists. If $\mathbf{b}$ were both of X-type and of Y-type, then necessarily

$$\mathbf{b} = \cdots XYXYXY \cdots,$$

and both $\rho^{-1}(\mathbf{b})$ and $\lambda^{-1}(\mathbf{b})$ would be of constant type. This is excluded by the hypothesis. Hence $\mathbf{b}^{(0)}$ has only one of the two possible desubstitutions. Let it be $\mathbf{b}^{(1)}$, so that $\mathbf{b}^{(0)} = \sigma_0(\mathbf{b}^{(1)})$. By Lemma 8.3.12, $\mathcal{S}(\mathbf{b}^{(1)}) \le 3$, and therefore $\mathbf{b}^{(1)}$ is of constant type, degenerate type, or regular type. By hypothesis it is not of constant type, and it cannot be of degenerate type, because then neither ρ^{-1} nor λ^{-1} could be continued. Thus $\mathbf{b}^{(1)}$ is again regular. Repeating this argument, every $\mathbf{b}^{(N)}$ is a regular word satisfying $\mathcal{S}(\mathbf{b}^{(N)}) \le 3$ and is of exactly one of the two types X or Y. In particular it contains both X and Y, and it contains the subword YX.

We first record the following fact. If a word contains a subword of the form w^*YXw, then applying λ produces a word containing

$$(\lambda(w)X)^*YX\lambda(w)X,$$

and applying ρ produces a word containing

$$(Y\rho(w))^*YXY\rho(w).$$

Indeed, since $\lambda(X) = X$ and $\lambda(Y) = XY$, the central YX is sent to $\lambda(YX) = XYX$, which again contains YX. On the right, an additional X appears before $\lambda(w)$, giving the right half $X\lambda(w)X$ in the appropriate position. On the left, it is enough to prove by induction on $|w|$ that

$$\lambda(w^*)X = (\lambda(w)X)^*.$$

For $|w| = 0$ both sides are X. If the assertion holds for words of length n, then

$$\lambda((wX)^*)X = \lambda(Xw^*)X = X\lambda(w^*)X = X(\lambda(w)X)^* = X(\lambda(wX))^* = (\lambda(wX)X)^*,$$
$$\lambda((wY)^*)X = \lambda(Yw^*)X = XY\lambda(w^*)X = XY(\lambda(w)X)^* = XYX\lambda(w)^* = X\lambda(Y)^*\lambda(w)^*$$
$$= X(\lambda(w)\lambda(Y))^* = X(\lambda(wY))^* = (\lambda(wY)X)^*.$$

This proves the induction step. The proof for ρ is similar, using

$$Y\rho(w^*) = (Y\rho(w))^*;$$

one verifies the assertion for Xw and Yw assuming it for w.

Now fix $N \geq 1$. Since $\mathbf{b}^{(N)}$ contains the subword YX, we regard it as $\emptyset^*YX\emptyset$. Applying $\sigma_{N-1}, \sigma_{N-2}, \ldots, \sigma_0$ successively and using the fact just proved, we find in $\mathbf{b} = \mathbf{b}^{(0)}$ a subword of the form $w_N^*YXw_N$. At each step w is replaced by either $\lambda(w)X$ or $Y\rho(w)$, so its length increases by at least 1. Therefore $|w_N| \geq N$. Thus $\mathbf{b}$ contains subwords of the form w^*YXw with $|w|$ arbitrarily large. Replacing X by $(2, 2)$ and Y by $(1, 1)$, we obtain arbitrarily long subwords of the form $w^*, 1, 1, 2, 2, w$. Hence, for arbitrarily long w, the word $\mathbf{b}$ can be cut in the form

$$\mathbf{b} = (\alpha^*, w^*, 1, 1, 2, 2, w, \beta)$$

with suitable one-sided infinite sequences α and β. At this cut,

$$\mathcal{S}(\mathbf{b}) \geq \ell(\alpha^*, w^*, 1, 1 \mid 2, 2, w, \beta) = [2; 2, w, \beta] + [0; 1, 1, w, \alpha].$$

As $|w| \to \infty$, the common finite part w in the two continued fractions becomes arbitrarily long, and therefore

$$[2; 2, w, \beta] + [0; 1, 1, w, \alpha] \longrightarrow 3.$$

Thus $\mathcal{S}(\mathbf{b}) \geq 3$. Since $\mathcal{S}(\mathbf{b}) \leq 3$ by hypothesis, we obtain $\mathcal{S}(\mathbf{b}) = 3$. ☐

PROPOSITION 8.3.15. *If $\mathcal{S}(\mathbf{b}) < 3$, then $\mathbf{b}$ is a mechanical word of rational slope.*

PROOF. Assume $\mathcal{S}(\mathbf{b}) < 3$. By Proposition 8.3.7, a degenerate word has value $\mathcal{S}(\mathbf{b}) = 3$, so $\mathbf{b}$ is not degenerate. Since $\mathcal{S}(\mathbf{b}) < 3$, we also have $\mathcal{S}(\mathbf{b}) \leq 3$. By Theorem 8.3.6, the word $\mathbf{b}$ is either of constant type or of regular type. If it is of constant type, then it is

$$\cdots XXX \cdots \quad \text{or} \quad \cdots YYY \cdots,$$

which are mechanical words of slopes 1 and ∞, respectively.

Assume that $\mathbf{b}$ is regular. Then one of $\rho^{-1}(\mathbf{b})$ and $\lambda^{-1}(\mathbf{b})$ is defined; call the defined word $\mathbf{b}^{(1)}$. If $\mathbf{b}^{(1)}$ is regular, repeat the same operation and define $\mathbf{b}^{(2)}$, continuing until a word $\mathbf{b}^{(n)}$ is no longer regular. If this process does not reach a constant word in finitely many steps, then there are two possibilities: either it reaches a degenerate word in finitely many steps, or it remains regular forever. These are the only possibilities because every word for which the operation is defined still satisfies $\mathcal{S} \leq 3$. In the first case, if a finite number of desubstitutions reaches a degenerate word, then the degenerate word contains symmetric subwords of the form w^*YXw with $|w|$ arbitrarily large. The same argument as in the proof of Lemma 8.3.14 then gives $\mathcal{S}(\mathbf{b}) = 3$, contradicting $\mathcal{S}(\mathbf{b}) < 3$.

In the second case, Lemma 8.3.14 again gives $\mathcal{S}(\mathbf{b}) = 3$, a contradiction. Hence $\mathbf{b}$ is obtained from a constant mechanical word by applying finitely many of λ and ρ. By Lemma 8.3.13, $\mathbf{b}$ is a mechanical word. Moreover, its slope is obtained from 1 or ∞ by applying finitely many times the transformations

$$t \mapsto 2 - \frac{1}{t}, \qquad t \mapsto t + 1.$$

Therefore the slope is rational. □

We now prove Markov's theorem.

Proof of Theorem 8.3.1. By Proposition 8.3.15, if $\mathcal{S}(\mathbf{b}) < 3$, then there exists $t \in [1, \infty] \cap \mathbb{Q}$ such that $\mathbf{b}$ is a mechanical word of slope t; more precisely, it is obtained after substituting $X \mapsto 2, 2$ and $Y \mapsto 1, 1$. Since t is rational, this mechanical word is unique up to shift. By Proposition 8.2.8, we therefore have

$$\mathbf{b} = (\dots, s(t), s(t), s(t), \dots).$$

Theorem 8.1.2 then gives

$$\mathcal{S}(\mathbf{b}) = \frac{\sqrt{(3m_t)^2 - 4}}{m_t}.$$

This proves $\mathcal{M} \cap (0, 3) \subset \mathcal{M}_{0,0,0}$, and hence Markov's theorem. □

4. Lagrange and Markov Constants from Lines of Irrational Slope

The argument in this section has three steps. We first associate finite sign blocks to rational-slope segments and recall that these blocks realize the discrete generalized Markov values. We then let the rational slopes converge to an irrational slope while keeping track of every finite block that occurs in the resulting bi-infinite sequence. Finally, we use the bi-infinite continued-fraction description of the Lagrange and Markov spectra to identify the limiting value. In the preceding section, in the case $(k_1, k_2, k_3) = (0, 0, 0)$, we showed that every bi-infinite sequence giving a Markov constant less than 3 is obtained from a mechanical word of rational slope. In this section we return to general (k_1, k_2, k_3, σ) and consider lines of irrational, rather than rational, slope. We determine the value obtained from such lines.

Throughout this section, fix $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$, and put $K := 3 + k_1 + k_2 + k_3$. We consider only positive slopes. This convention agrees with the definition, in Chapter 7, of the generalized strongly admissible sequence $s(t)$ for $t \in [0, \infty] \cap \mathbb{Q}$. A line of positive slope will be oriented in the direction in which the x-coordinate increases. Reversing the orientation only reverses the resulting sequence, and therefore does not change the value of $\mathcal{S}$.

Definition 8.4.1. Let l be an oriented line of positive irrational slope, and assume that l does not pass through any point of $\widetilde{\mathbb{R}^2}$. Apply the triangle-crossing rule and the edge-crossing rule to l, and write down the signs in the order in which l crosses the corresponding triangles and edges. In the resulting bi-infinite sign sequence, record the lengths of the successive maximal blocks of equal signs. We denote the resulting bi-infinite sequence of positive integers by

$$\mathbf{b}(l) = (b_n)_{n \in \mathbb{Z}}.$$

The origin of the index is chosen arbitrarily; thus $\mathbf{b}(l)$ is determined only up to shift.

Since $\mathcal{S}(\mathbf{b})$ is invariant under shifts, the arbitrary choice of the index origin in Definition 8.4.1 causes no ambiguity. The goal of this section is to prove the following theorem.

Theorem 8.4.2. *Fix $(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0}$ and $\sigma \in \mathfrak{S}_3$, and put $K = 3 + k_1 + k_2 + k_3$. Let l be a line of positive irrational slope which does not pass through any point of $\widetilde{\mathbb{R}^2}$. Then*

$$\mathcal{S}(\mathbf{b}(l)) = K.$$

For the proof we prepare four lemmas. The first one says that the quantity ℓ_n used in the definition of $\mathcal{S}$ can be approximated, to arbitrary accuracy, from a sufficiently large finite block.

LEMMA 8.4.3. *For every $\varepsilon > 0$, there exists $N \geq 1$ with the following property. If two bi-infinite sequences of positive integers $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ and $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$ satisfy*

$$a_i = c_i \qquad (-N \leq i \leq N),$$

then

$$|\ell_0(\mathbf{a}) - \ell_0(\mathbf{c})| < \varepsilon.$$

PROOF. By Lemma 3.1.7, for $N \geq 2$ we have

$$|[a_0; a_1, a_2, \ldots] - [c_0; c_1, c_2, \ldots]| < \frac{1}{N(N-1)}.$$

The same argument, applied to the continued fractions on the left side of the divider, gives

$$|[0; a_{-1}, a_{-2}, \ldots] - [0; c_{-1}, c_{-2}, \ldots]| < \frac{1}{N(N-1)}.$$

Therefore

$$|\ell_0(\mathbf{a}) - \ell_0(\mathbf{c})| < \frac{2}{N(N-1)}.$$

Taking N sufficiently large makes the right-hand side smaller than ε. □

The next lemma says that every finite block arising from a line of irrational slope can be approximated by a generalized strongly admissible sequence of rational slope.

LEMMA 8.4.4. *Let l be a line of positive irrational slope which does not pass through any point of $\widetilde{\mathbb{R}^2}$. Then every finite block appearing in $\mathbf{b}(l)$ also appears in the periodic sequence*

$$^{\infty}s(t)^{\infty} = (\ldots, s(t), s(t), s(t), \ldots)$$

for some reduced fraction $t \in (0, \infty) \cap \mathbb{Q}$ with sufficiently large denominator.

PROOF. Fix a finite block of $\mathbf{b}(l)$. This block is determined by a finite segment of l, namely by the order in which that segment crosses the triangles and edges of $\widetilde{\mathbb{R}^2}$ and by the signs assigned to them. Since l does not pass through any point of $\widetilde{\mathbb{R}^2}$, this finite segment has positive distance from the relevant points. Therefore the crossing order and the assigned signs in this finite region remain unchanged if the slope and intercept of the line are changed sufficiently slightly.

Write the slope and intercept of l as τ and θ, respectively. Choose a reduced rational number $t = p/q$ sufficiently close to τ. If q is sufficiently large, then integer translates of the line L_t used to define $s(t)$ have intercepts which approximate θ arbitrarily well at mesh size $1/q$. Indeed, translating L_t by an integer vector (m, n) changes the intercept by $n - tm$, and, since $t = p/q$ with $(p, q) = 1$, the possible changes run through $(1/q)\mathbb{Z}$. Hence, for sufficiently large q, we may choose an integer translate of L_t whose intercept is sufficiently close to θ.

In the fixed finite region, this translated line gives the same sign sequence as l. On the other hand, integer translations preserve the sign rules on $\widetilde{\mathbb{R}^2}$, and therefore the bi-infinite sequence obtained from this translated line is a shift of $^{\infty}s(t)^{\infty}$. Thus the fixed finite block also appears in $^{\infty}s(t)^{\infty}$. □

The following lemma states that, for regular lines of the same irrational slope, the set of finite blocks is independent of the intercept.

LEMMA 8.4.5. *Let l and l' be two lines of the same positive irrational slope, and assume that neither of them passes through any point of $\widetilde{\mathbb{R}^2}$. Then the set of finite blocks appearing in $\mathbf{b}(l)$ coincides with the set of finite blocks appearing in $\mathbf{b}(l')$.*

PROOF. Let the common slope be $\tau \notin \mathbb{Q}$. Translating l by an integer vector (m, n) changes its intercept by $n - \tau m$. Since τ is irrational, Theorem A.4.1, applied to $-\tau$, shows that

$$\{n - \tau m \mid m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}/\mathbb{Z}$.

Fix a finite block appearing in $\mathbf{b}(l)$. Since l does not pass through any point of $\widetilde{\mathbb{R}^2}$, the finite segment of l which gives this block is stable under sufficiently small translations. By the density above, some integer translate of l can be made sufficiently close to l' in the relevant finite region. Hence, in that finite region, the two lines give the same sign sequence and therefore the same integer block. Since integer translations preserve the sign rules on $\widetilde{\mathbb{R}^2}$, this block also appears in $\mathbf{b}(l')$. The reverse inclusion is proved in the same way. $\square$

Finally, we recall the values obtained from rational slopes.

LEMMA 8.4.6. *For every positive reduced fraction $t \in (0, \infty) \cap \mathbb{Q}$, one has*

$$\mathcal{S}({}^{\infty}s(t)^{\infty}) < K.$$

Moreover, if $(t_j)_{j\geq 0}$ is a sequence of pairwise distinct positive reduced fractions converging to an irrational number τ, then

$$\lim_{j\to\infty} \mathcal{S}({}^{\infty}s(t_j)^{\infty}) = K.$$

PROOF. By Theorems 8.1.1 and 3.3.3, if (m_t, i_t) denotes the (k_1, k_2, k_3, σ)-GM number and its position corresponding to t, then

$$\mathcal{S}({}^{\infty}s(t)^{\infty}) = \frac{\sqrt{(Km_t - k_t)^2 - 4}}{m_t}.$$

The right-hand side is strictly smaller than K.

Now suppose that $t_j \to \tau \notin \mathbb{Q}$. If $t_j = p_j/q_j$ is written in lowest terms, then $p_j + q_j \to \infty$. Hence the number of triangles and edges crossed by L_{t_j} tends to infinity, and the length of the corresponding generalized strongly admissible sequence $s(t_j)$ also tends to infinity. By Theorem 7.4.4, m_{t_j} is the $(2,1)$-entry of $F_{s(t_j)}$. Since the entries of $s(t_j)$ are positive integers and the lengths of these sequences tend to infinity, we have $m_{t_j} \to \infty$. Since each k_{t_j} is one of the fixed integers k_1, k_2, k_3, the sequence $(k_{t_j})_j$ is bounded. Therefore

$$\frac{\sqrt{(Km_{t_j} - k_{t_j})^2 - 4}}{m_{t_j}} = \sqrt{\left(K - \frac{k_{t_j}}{m_{t_j}}\right)^2 - \frac{4}{m_{t_j}^2}} \longrightarrow K.$$

$\square$

We now prove the theorem.

PROOF OF THEOREM 8.4.2. First we prove $\mathcal{S}(\mathbf{b}(l)) \leq K$. Fix $r \in \mathbb{Z}$. It is enough to show that $\ell_r(\mathbf{b}(l)) \leq K$.

Let $\varepsilon > 0$. By Lemma 8.4.3, if N is sufficiently large, then the value of ℓ_r is determined up to an error smaller than ε by the central finite block

$$b_{r-N}, \ldots, b_r, \ldots, b_{r+N}.$$

By Lemma 8.4.4, this finite block also appears in the periodic sequence ${}^{\infty}s(t)^{\infty}$ for some rational slope t. Hence, for some position j, we have

$$\ell_r(\mathbf{b}(l)) \leq \ell_j({}^{\infty}s(t)^{\infty}) + \varepsilon.$$

By Lemma 8.4.6, it follows that

$$\ell_r(\mathbf{b}(l)) \leq \mathcal{S}({}^{\infty}s(t)^{\infty}) + \varepsilon < K + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\ell_r(\mathbf{b}(l)) \leq K$. Since r was arbitrary, we obtain

$$\mathcal{S}(\mathbf{b}(l)) \leq K.$$

Next we prove $\mathcal{S}(\mathbf{b}(l)) \geq K$. Let τ be the slope of l, and choose a sequence of pairwise distinct reduced fractions $t_j \in (0, \infty) \cap \mathbb{Q}$ converging to τ. By Lemma 8.4.6,

$$\mathcal{S}({}^{\infty}s(t_j)^{\infty}) \longrightarrow K.$$

For each j, choose a position in the periodic sequence ${}^{\infty}s(t_j)^{\infty}$ at which $\mathcal{S}({}^{\infty}s(t_j)^{\infty})$ is attained, and shift the sequence so that this position becomes 0. Such a position exists because a periodic sequence has only finitely many candidates for the value of $\mathcal{S}$. Denote the shifted sequence by $\mathbf{c}^{(j)}$. Thus $\ell_0(\mathbf{c}^{(j)}) = \mathcal{S}({}^{\infty}s(t_j)^{\infty})$.

First we check that the entries of the sequences $\mathbf{c}^{(j)}$ are uniformly bounded. For every $n \in \mathbb{Z}$,

$$c_n^{(j)} < \ell_n(\mathbf{c}^{(j)}) \leq \mathcal{S}(\mathbf{c}^{(j)}) < K.$$

Since $K = 3 + k_1 + k_2 + k_3$ is an integer and $c_n^{(j)}$ is a positive integer, we have

$$c_n^{(j)} \in \{1, 2, \ldots, K-1\}$$

for all j and n. Order the integers as $0, 1, -1, 2, -2, \ldots$. Looking first at the coordinate 0, the sequence $(c_0^{(j)})_j$ takes values in the finite set $\{1, \ldots, K-1\}$, so some value occurs infinitely often. Passing to a subsequence, we may assume that $c_0^{(j)}$ is constant. Then, inside this subsequence, pass to a further subsequence on which $c_1^{(j)}$ is constant. Repeating this process along the order $0, 1, -1, 2, -2, \ldots$ and finally taking a diagonal subsequence, we may assume that, for each fixed $n \in \mathbb{Z}$, the value $c_n^{(j)}$ is eventually constant. Hence

$$c_n := \lim_{j \to \infty} c_n^{(j)}$$

exists for every $n \in \mathbb{Z}$. In what follows, we rename this diagonal subsequence as $\mathbf{c}^{(j)}$ and put $\mathbf{c} = (c_n)_{n \in \mathbb{Z}}$.

We next prove that $\lim_{j \to \infty} \ell_0(\mathbf{c}^{(j)}) = \ell_0(\mathbf{c})$. Let $\varepsilon > 0$. By Lemma 8.4.3, there exists $N \geq 1$ such that, whenever two bi-infinite sequences of positive integers agree on the central finite block $[-N, N]$, the corresponding values of ℓ_0 differ by less than ε. For each n, we have $c_n^{(j)} \to c_n$, and the entries are integer-valued. Therefore, for the finitely many coordinates $-N, \ldots, N$, we have, for all sufficiently large j,

$$c_n^{(j)} = c_n \qquad (-N \leq n \leq N)$$

simultaneously. Hence, for all sufficiently large j, $|\ell_0(\mathbf{c}^{(j)}) - \ell_0(\mathbf{c})| < \varepsilon$. Thus $\ell_0(\mathbf{c}^{(j)}) \to \ell_0(\mathbf{c})$.

On the other hand, the sequences $\mathbf{c}^{(j)}$ were chosen so that $\ell_0(\mathbf{c}^{(j)}) \to K$. Passing to a subsequence does not change this limit. Hence, by uniqueness of limits, $\ell_0(\mathbf{c}) = K$.

We now show that every finite block of $\mathbf{c}$ also appears in $\mathbf{b}(l)$. Fix a finite block

$$W = (c_{-N}, c_{-N+1}, \ldots, c_N).$$

By the construction of the diagonal subsequence, for all sufficiently large j we have

$$(c_{-N}^{(j)}, c_{-N+1}^{(j)}, \ldots, c_N^{(j)}) = W.$$

Therefore W appears, for infinitely many j, in the periodic sequence ${}^{\infty}s(t_j)^{\infty}$ arising from a rational line of slope t_j.

For each such j, let γ_j be a finite line segment of slope t_j from which the block W is read. The triangulation and the sign rules are invariant under integer translations. Hence translating γ_j by an integer vector does not change the finite block read from it. We translate each γ_j so that its midpoint lies in the fundamental square $[0,1]^2$.

By this uniform boundedness of the midpoints, after passing to a subsequence we may assume that the midpoints of γ_j converge to a point $P \in [0,1]^2$. Since W is a fixed finite block and $t_j \to \tau$, the lengths of the segments γ_j are uniformly bounded for all sufficiently large j. Therefore, after passing to a further subsequence, we may assume that the segments γ_j converge to a finite line segment γ of slope τ passing through P.

It may happen that γ passes through a point of $\widetilde{\mathbb{R}^2}$. In that case, locally the limiting segment still contains the same finite block W as the segments γ_j, but the whole line containing γ does not define a bi-infinite sequence because it hits a singular point. We therefore have to move the line slightly, by a parallel translation with the same slope τ, so that it avoids the points of $\widetilde{\mathbb{R}^2}$ while still containing the same finite block W.

Move the intercept continuously and sufficiently slightly, keeping the slope equal to τ, in such a way that the part of the line which determines the block W does not change. The set of intercepts obtained in this way contains a nonempty open interval I. For every $\theta' \in I$, the line $l_{\theta'}$ of slope τ reads the finite block W. We choose θ' so that the line $l_{\theta'} : y = \tau x + \theta'$ does not pass through any point of $\widetilde{\mathbb{R}^2}$. Indeed, the condition that l_θ pass through a point (a, b) of $\widetilde{\mathbb{R}^2}$ is

$\theta = b - \tau a$. Thus the set of intercepts for which a line of slope τ passes through a point of $\widetilde{\mathbb{R}^2}$ is the countable set

$$E_\tau = \{\, b - \tau a \mid (a,b) \text{ is a point of } \widetilde{\mathbb{R}^2}\,\}.$$

Since removing the countable set $I \cap E_\tau$ from the nonempty open interval I cannot make it empty, we can choose $\theta' \in I \setminus (I \cap E_\tau)$. Then $l_{\theta'}$ does not pass through any point of $\widetilde{\mathbb{R}^2}$. Moreover, because $\theta' \in I$, the sequence obtained from $l_{\theta'}$ contains the finite block W.

The originally fixed line l also has slope τ and does not pass through any point of $\widetilde{\mathbb{R}^2}$. Therefore, by Lemma 8.4.5, the set of finite blocks arising from regular lines of the same irrational slope is independent of the intercept. Hence W also appears in $\mathbf{b}(l)$.

Now let $\varepsilon > 0$. By Lemma 8.4.3, if N is sufficiently large, then the value of ℓ_0 is determined up to an error smaller than ε by the central block $W = (c_{-N}, \ldots, c_0, \ldots, c_N)$. This finite block appears in $\mathbf{b}(l)$, so for some position q we have

$$\ell_q(\mathbf{b}(l)) > \ell_0(\mathbf{c}) - \varepsilon = K - \varepsilon.$$

Therefore $\mathcal{S}(\mathbf{b}(l)) \geq K - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $\mathcal{S}(\mathbf{b}(l)) \geq K$.

Combining the two inequalities, we obtain

$$\mathcal{S}(\mathbf{b}(l)) = K = 3 + k_1 + k_2 + k_3.$$

□

REMARK 8.4.7. For $(k_1, k_2, k_3) = (0,0,0)$, Theorem 8.4.2 corresponds to the fact that a bi-infinite sequence obtained from an irrational-slope mechanical word by substituting $X \mapsto (2,2)$ and $Y \mapsto (1,1)$ gives the boundary value 3. Thus the theorem may be viewed as a sign-rule formulation of the statement that the accumulation point of the discrete values arising from rational-slope generalized strongly admissible sequences is $3 + k_1 + k_2 + k_3$.

We close this section by recording the corresponding Markov and Lagrange-constant realization of the same value.

COROLLARY 8.4.8. *With the notation of Theorem 8.4.2, define*

$$\theta_l := [b_0; b_1, b_2, \ldots], \qquad \eta_l := [b_{-1}; b_{-2}, b_{-3}, \ldots],$$

and put

$$Q_{\mathbf{b}(l)}(x,y) := (x - \theta_l y)\left(x + \frac{1}{\eta_l} y\right).$$

Then $Q_{\mathbf{b}(l)} \in \mathcal{R}$ *and*

$$\mathcal{M}(Q_{\mathbf{b}(l)}) = K.$$

PROOF. Since all entries b_n are positive integers, we have $\theta_l > 1$ and $\eta_l > 1$. The two roots of $Q_{\mathbf{b}(l)}(x,1)$ are therefore θ_l and $-1/\eta_l$, where $\theta_l > 1$ and $-1 < -1/\eta_l < 0$. Hence $Q_{\mathbf{b}(l)}$ is canonical reduced, so $Q_{\mathbf{b}(l)} \in \mathcal{R}$. By Theorem 4.3.3, the bi-infinite continued-fraction sequence attached to $Q_{\mathbf{b}(l)}$ is exactly $\mathbf{b}(l)$. Thus

$$\mathcal{M}(Q_{\mathbf{b}(l)}) = \mathcal{S}(\mathbf{b}(l)).$$

Theorem 8.4.2 gives $\mathcal{S}(\mathbf{b}(l)) = K$, and hence $\mathcal{M}(Q_{\mathbf{b}(l)}) = K$. □

COROLLARY 8.4.9. *With the notation of Theorem 8.4.2, for any* $r \in \mathbb{Z}$, *put*

$$\alpha_r := [0; b_r, b_{r+1}, b_{r+2}, \ldots].$$

Then

$$\mathcal{L}(\alpha_r) = K.$$

PROOF. It is enough to prove the case $r = 0$, since the other cases are obtained by shifting the indices. Put

$$\alpha := [0; b_0, b_1, b_2, \ldots].$$

By Theorem 3.1.3,

$$\mathcal{L}(\alpha) = \limsup_{n\to\infty} \left([b_n; b_{n+1}, b_{n+2}, \ldots] + [0; b_{n-1}, b_{n-2}, \ldots, b_0]\right).$$

On the other hand, for the bi-infinite sequence $\mathbf{b}(l)$ we have

$$\ell_n(\mathbf{b}(l)) = [b_n; b_{n+1}, b_{n+2}, \ldots] + [0; b_{n-1}, b_{n-2}, \ldots].$$

Compare the two continued fractions

$$[0; b_{n-1}, b_{n-2}, \ldots, b_0] \quad \text{and} \quad [0; b_{n-1}, b_{n-2}, \ldots].$$

They have the same first n partial quotients. Hence Lemma 3.1.7 gives

$$|[0; b_{n-1}, b_{n-2}, \ldots, b_0] - [0; b_{n-1}, b_{n-2}, \ldots]| \longrightarrow 0 \qquad (n \to \infty).$$

Therefore

$$\mathcal{L}(\alpha) = \limsup_{n\to\infty} \ell_n(\mathbf{b}(l)).$$

By Theorem 8.4.2, $\mathcal{S}(\mathbf{b}(l)) = K$. Hence $\ell_n(\mathbf{b}(l)) \le K$ for all $n \in \mathbb{Z}$, and so

$$\mathcal{L}(\alpha) = \limsup_{n\to\infty} \ell_n(\mathbf{b}(l)) \le K.$$

We prove the reverse inequality. Let $\varepsilon > 0$. Since $\mathcal{S}(\mathbf{b}(l)) = K$, there exists $r_0 \in \mathbb{Z}$ such that

$$\ell_{r_0}(\mathbf{b}(l)) > K - \varepsilon.$$

By Lemma 8.4.3, if $N \ge 1$ is sufficiently large, then any bi-infinite sequence of positive integers whose central block agrees with

$$W := (b_{r_0-N}, b_{r_0-N+1}, \ldots, b_{r_0+N})$$

has, at the corresponding central position, an ℓ-value which differs from $\ell_{r_0}(\mathbf{b}(l))$ by less than ε.

We now show that the finite block W appears in the positive direction in $\mathbf{b}(l)$ infinitely many times. Only finitely many crossings in a finite part of the line l are needed to read W. Since l does not pass through any point of $\widetilde{\mathbb{R}^2}$, this finite part has positive distance from the relevant lattice points and the boundaries involved in the sign rules. Thus sufficiently small parallel translations of l do not change the crossing order or the signs in this finite part. Equivalently, the set of positions of lines of slope τ which read the same finite block W contains a nonempty open subset U of $\mathbb{R}^2/\mathbb{Z}^2$.

Moving along the line l in the positive direction corresponds, on $\mathbb{R}^2/\mathbb{Z}^2$, to moving along the positive orbit of a line of slope τ. Since τ is irrational, this positive orbit is dense in $\mathbb{R}^2/\mathbb{Z}^2$. Therefore it enters U infinitely many times. This means that the finite block W occurs in the positive direction in $\mathbf{b}(l)$ infinitely often. Hence we can choose positions

$$q_1 < q_2 < q_3 < \cdots$$

at which W occurs as the central block. For every i, the choice of N gives

$$\ell_{q_i}(\mathbf{b}(l)) > K - 2\varepsilon.$$

Consequently,

$$\limsup_{n\to\infty} \ell_n(\mathbf{b}(l)) \ge K - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary,

$$\limsup_{n\to\infty} \ell_n(\mathbf{b}(l)) \ge K.$$

Together with the opposite inequality, this proves

$$\mathcal{L}(\alpha) = \limsup_{n\to\infty} \ell_n(\mathbf{b}(l)) = K.$$

□

5. The Relation Between the $(0,0,0)$-Type and the $(2,2,2)$-Type

Among GM numbers, there is a special relation between the $(0,0,0)$-type, namely the ordinary Markov numbers, and the $(2,2,2)$-type. We now describe it. Consider the case $k_1 = k_2 = k_3 = 2$, that is, the equation

$$x^2 + y^2 + z^2 + 2yz + 2zx + 2xy = 9xyz \tag{8.5.1}$$

We have the following theorem.

THEOREM 8.5.1. *Let (a,b,c) be a positive integer solution of the Markov equation. Then (a^2,b^2,c^2) is a solution of the $(2,2,2)$-GM equation. Conversely, if a positive integer triple (A,B,C) is a solution of the $(2,2,2)$-GM equation, then $(\sqrt{A},\sqrt{B},\sqrt{C})$ is a solution of the Markov equation.*

PROOF. We prove the first assertion. The case $(a,b,c) = (1,1,1)$ is clear. Assume that (a,b,c) is an integer solution of the Markov equation. It suffices to show that the Vieta jumps of (a^2,b^2,c^2) in (8.5.1) are

$$\left(\left(\frac{b^2+c^2}{a}\right)^2, b^2, c^2\right), \quad \left(a^2, \left(\frac{a^2+c^2}{b}\right)^2, c^2\right), \quad \left(a^2, b^2, \left(\frac{a^2+b^2}{c}\right)^2\right)$$

We prove only the first Vieta jump.

The first Vieta jump of (a^2,b^2,c^2) in (8.5.1) is

$$\left(\frac{(b^2)^2 + 2b^2c^2 + (c^2)^2}{a^2}, b^2, c^2\right) = \left(\left(\frac{b^2+c^2}{a}\right)^2, b^2, c^2\right)$$

which has the desired form.

We next prove the converse. By the first assertion and Theorem 5.1.3, every positive integer solution of (8.5.1) can be written inductively in the form (a^2,b^2,c^2), where (a,b,c) is a solution of the Markov equation. The assertion follows. □

This relation gives a direct comparison between the discrete Markov spectrum and the $(2,2,2)$-generalized discrete Markov spectrum.

THEOREM 8.5.2. *If $r \in \mathcal{M}_{0,0,0}$, then $3r \in \mathcal{M}_{2,2,2}$. Conversely, if $R \in \mathcal{M}_{2,2,2}$, then $R/3 \in \mathcal{M}_{0,0,0}$.*

PROOF. We first prove the first assertion. If $r \in \mathcal{M}_{0,0,0}$, then there exists a Markov number m such that $r = \frac{\sqrt{9m^2-4}}{m}$. Hence $3r = \frac{3\sqrt{9m^2-4}}{m} = \frac{\sqrt{81m^4-36m^2}}{m^2} = \frac{\sqrt{(9m^2-2)^2-4}}{m^2}$. By Theorem 8.5.1, m^2 is a $(2,2,2)$-GM number. Thus $3r \in \mathcal{M}_{2,2,2}$.

The converse follows by reversing the same calculation. □

6. Frobenius's Uniqueness Conjecture and Its Generalization

In this section we discuss natural generalizations of Frobenius's uniqueness conjecture. The original conjecture is the following.

CONJECTURE 8.6.1 ([**Fro13**]). *For every Markov number c, there is a unique Markov triple (a,b,c) with $a \le b \le c$.*

In terms of the Markov tree and fraction labels, this can be reformulated as follows.

CONJECTURE 8.6.2. *Fix $\sigma \in \mathfrak{S}_3$ and consider $\mathrm{M}\mathbb{T}(0,0,0,\sigma)$. Let $t,s \in [1,\infty]$ be reduced fractions, and let m_t, m_s be the Markov numbers whose fraction labels are t and s, respectively. If $m_t = m_s$, then $t = s$.*

Behind this elementary-looking conjecture lies the following equivalent-looking spectral formulation.

CONJECTURE 8.6.3. *For every $L \in \mathcal{M}_{0,0,0}$, if $L = \mathcal{L}(\alpha) = \mathcal{L}(\beta)$, then α and β are $GL(2,\mathbb{Z})$-equivalent.*

In other words, the irrational numbers that give Lagrange spectrum values less than 3 should be unique up to $GL(2,\mathbb{Z})$-equivalence. Using Markov's theorem, one can show that these two conjectures are equivalent.

PROPOSITION 8.6.4. *Conjecture 8.6.1 (equivalently, Conjecture 8.6.2) and Conjecture 8.6.3 are equivalent.*

PROOF. Assume Conjecture 8.6.2. Suppose that there exist α, β with $L = \mathcal{L}(\alpha) = \mathcal{L}(\beta)$ but $\alpha \not\sim \beta$. Since $L \in \mathcal{M}_{0,0,0}$, we have $L < 3$. Therefore the bi-infinite sequences giving L correspond, by Proposition 8.3.15, to mechanical words of rational slope. Let $\mathbf{b}_t$ and $\mathbf{b}_s$ be the mechanical words corresponding to α and β, respectively, obtained through the construction in Corollary 3.3.4. By Theorem 2.4.6, the assumption $\alpha \not\sim \beta$ implies that these two sequences do not agree even up to cyclic shift. Hence $t \neq s$. By Conjecture 8.6.2, $m_t \neq m_s$, while

$$\mathcal{L}(\alpha) = \mathcal{S}(\mathbf{b}_t) = \frac{\sqrt{9m_t^2 - 4}}{m_t}, \quad \mathcal{L}(\beta) = \mathcal{S}(\mathbf{b}_s) = \frac{\sqrt{9m_s^2 - 4}}{m_s}. \tag{8.6.1}$$

Thus $\mathcal{L}(\alpha) \neq \mathcal{L}(\beta)$, a contradiction. Therefore $\alpha \sim \beta$.

Conversely, assume Conjecture 8.6.3. Let $t, s \in [1, \infty] \cap \mathbb{Q}$ with $t \neq s$. Let $\mathbf{b}_t$ and $\mathbf{b}_s$ be mechanical words of slopes t and s, respectively. By Proposition 8.2.7, $\mathbf{b}_t \neq \mathbf{b}_s$; more precisely, if $t \neq s$, then the corresponding periodic sequences do not agree even up to cyclic shift. Assuming Conjecture 8.6.3, the quadratic irrationals α and β obtained from $\mathbf{b}_t$ and $\mathbf{b}_s$ by Corollary 3.3.4 must satisfy $\mathcal{L}(\alpha) \neq \mathcal{L}(\beta)$. Since (8.6.1) again holds, it follows that $m_t \neq m_s$. This proves the assertion. □

As a generalization of Conjecture 8.6.1, Gyoda and Matsushita [**GM23a**] proposed the following problem.

PROBLEM 8.6.5. *For every (k_1, k_2, k_3)-GM number c, is there a unique (k_1, k_2, k_3)-GM triple (a, b, c) with $a \leq b \leq c$?*

Similarly, as a generalization of Conjecture 8.6.3, one may consider the following problem.

PROBLEM 8.6.6. *For every $L \in \mathcal{M}_{k_1,k_2,k_3}$, if $L = \mathcal{L}(\alpha) = \mathcal{L}(\beta)$, then are α and β $GL(2,\mathbb{Z})$-equivalent?*

Note that Problems 8.6.5 and 8.6.6 are not equivalent.

Counterexamples to Problem 8.6.5 occur when k_1, k_2, k_3 are pairwise distinct. For instance, $(1, 81, 17)$ and $(7, 81, 2)$ are both positive integer solutions of the $(1, 2, 0)$-GM equation.

The uniqueness phenomenon below 3 is special. Hurwitz already observed that beyond the classical Markov part one can have different quadratic irrationals realizing the same value of the Markov spectrum [**Hur91**]; in the notation of this text, this means that Problem 8.6.6 fails for values outside $\mathcal{M}_{0,0,0}$. We give here concrete examples of this phenomenon. When not all of k_1, k_2, k_3 are zero, there is an evident source of counterexamples coming from Theorem 8.1.1. For a generalized strongly admissible sequence $s(t)$, set $\alpha = [\overline{s(t)}]$ and $\beta = [\overline{s^*(1/t)}]$. Then Theorem 8.1.1 gives $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$.

Moreover, by Remark 7.4.3 (7), the periodic parts of $\overline{s(t)}$ and $\overline{s^*(1/t)}$ do not in general agree even up to cyclic shift, except in the case $k_1 = k_2 = k_3 = 0$. Since, by Theorem 2.4.6, two irrational numbers are $GL(2,\mathbb{Z})$-equivalent if and only if their periodic parts agree up to cyclic shift, the numbers α and β are in general not $GL(2,\mathbb{Z})$-equivalent.

There are also counterexamples of a different nature. They can be constructed from counterexamples to Problem 8.6.5. Put $L = \frac{\sqrt{((3+0+1+2)81-2)^2-4}}{81} = \frac{2\sqrt{723}}{9}$. Then there exist two different quadratic irrationals α, β such that $L = \mathcal{L}(\alpha) = \mathcal{L}(\beta)$. One corresponds to the $(1, 2, 0)$-GM triple $(1, 81, 17)$, and the other corresponds to $(7, 81, 2)$.

The triple $(1, 81, 17)$ corresponds to a vertex of $\mathrm{M}\mathbb{T}(1, 2, 0, \mathrm{id})$, and the fraction label of 81 is $1/3$. The corresponding quadratic irrational is $\alpha = [\overline{5, 1, 3, 3, 1, 4}] = \frac{\sqrt{723}+25}{9}$. On the other hand, $(7, 81, 2)$ corresponds to a vertex of $\mathrm{M}\mathbb{T}(1, 2, 0, (1\ 2\ 3))$, and the fraction label is $2/3$. The corresponding quadratic irrational is $\beta = [\overline{5, 1, 1, 5, 3, 2}] = \frac{\sqrt{723}+23}{9}$. The two periodic parts are not equal, nor are they related by cyclic shift or reversal. Hence this equality is not the trivial equality coming from the same periodic continued fraction; it comes from GM numbers belonging to different GM trees that yield the same value.

The triples $(1, 81, 17)$ and $(7, 81, 2)$ arise from different GM trees: the former has $\sigma = \mathrm{id}$, while the latter has $\sigma = (1\,2\,3)$. This suggests that, when formulating a generalized Frobenius uniqueness conjecture, the appropriate object to generalize is the injectivity formulation, Conjecture 8.6.2.

CONJECTURE 8.6.7. *Fix* $\sigma \in \mathfrak{S}_3$ *and consider* $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$. *For reduced fractions* $t, s \in [0, 1]$, *let* m_t, m_s *be the* (k_1, k_2, k_3)*-GM numbers whose fraction labels are* t *and* s, *respectively. If* $m_t = m_s$, *then* $t = s$.

The interval is restricted to $[0, 1]$ because the labels on $[1, \infty]$ can be treated equivalently by using the reversal $t \mapsto 1/t$ and the replacement of σ by σ^* appearing in Theorem 5.2.10.

When $k_1 = k_2 = k_3$, the tree $\mathrm{M}\mathbb{T}(k_1, k_2, k_3, \sigma)$ is essentially independent of σ. In this case the conjecture is equivalent to giving an affirmative answer to Problem 8.6.5 in the case $k_1 = k_2 = k_3$. When k_1, k_2, k_3 are pairwise distinct, however, the tree depends on σ, and the conjecture is no longer equivalent to Problem 8.6.5. At present, no counterexample to this conjecture is known.

CHAPTER 9

Further Topics

This chapter collects several directions related to the Lagrange spectrum, the Markov spectrum, generalized Markov numbers, generalized Cohn matrices, and the generalized discrete Markov spectrum. These topics may at first appear rather specialized, but they touch many areas: Diophantine approximation, continued fractions, combinatorics on words, hyperbolic geometry, cluster algebras from surfaces, arithmetic geometry, and dynamical systems. The list is not meant to be exhaustive, and it deliberately overlaps with some of the historical discussion in Chapter 1. Some of the topics lie outside the author's own area of expertise, and some recent works cited here are still preprints; the purpose is only to provide entry points for further reading.

(1) **Hall's ray, Freiman's constant, and the transition region**
Between the discrete part below 3, which is governed by Markov's theorem, and the region where Hall's ray begins, lies one of the most complicated parts of the spectrum. Hall proved that $[6,\infty)$ is contained in the Lagrange spectrum, and Freiman determined the initial point c_F of the largest half-line contained in it [**Hal47, Fre75**]. Thus a natural next problem is to understand the transition region $[3, c_F)$. The description by bi-infinite continued fractions studied in Chapters 3 and 4 remains one of the basic tools for this purpose. Standard references include the monograph of Cusick–Flahive and the more recent dynamical and fractal account of Lima–Matheus–Moreira–Romana [**CF89, LMMR20**].

(2) **Where do $\mathcal{L}$ and $\mathcal{M}$ begin to differ?**
This text used the inclusion $\mathcal{L} \subset \mathcal{M}$ and the equality below 3. A natural question is how large $\mathcal{M} \setminus \mathcal{L}$ is and how close to 3 it begins to appear. Moreira proved that, for every half-line $(-\infty, t)$, the intersections of the Lagrange and Markov spectra with this half-line have the same Hausdorff dimension [**Mor18**]. On the other hand, Erazo–Lima–Matheus–Moreira–Vieira proved that $\inf(\mathcal{M} \setminus \mathcal{L}) = 3$, showing that the two spectra already differ immediately above 3 [**ELM$^+$24**]. Thus they are extremely close from the point of view of dimension, but separate at once as sets.

(3) **Dynamical Lagrange and Markov spectra**
Classically, $\mathcal{L}$ and $\mathcal{M}$ can be expressed through the limsup and supremum of the functions $\ell_n = \ell_0 \circ \sigma^n$, where σ is the shift on bi-infinite continued-fraction sequences and ℓ_0 is the sum of the two continued fractions cut at the origin. Replacing the shift space by a general set X, the shift by a self-map $\phi\colon X \to X$, and ℓ_0 by a function $f\colon X \to \mathbb{R}$ (or $\mathbb{C}$), one obtains dynamical analogues by considering the limsup and supremum of $f(\phi^n(x))$. These are called dynamical Lagrange and Markov spectra. Cerqueira–Matheus–Moreira studied such spectra for horseshoes of area-preserving surface maps and proved continuity of Hausdorff dimension and equality of the dimensions of the Lagrange-type and Markov-type sets [**CMM18**]. Cerqueira–Moreira–Romana treated related questions for geodesic flows on negatively curved surfaces [**CMR22**].

(4) **Lagrange spectra of translation surfaces**
The classical Lagrange spectrum also has a geometric interpretation in terms of how deeply a geodesic on the modular surface enters a cusp. Starting from this interpretation, one can define analogous spectra in Teichmüller dynamics. Hubert–Marchese–Ulcigrai introduced Lagrange spectra for closed $SL(2,\mathbb{R})$-invariant loci in the moduli space of translation surfaces [**HMU15**]. Artigiani–Marchese–Ulcigrai then proved that the Lagrange spectrum of a Veech surface has a Hall ray [**AMU16**]. Thus there are meaningful analogues of the Lagrange spectrum beyond the modular surface.

(5) **A multiplicative analogue of the Lagrange spectrum**
The classical Lagrange spectrum is related to approximation properties of the arithmetic progression $n\alpha$ modulo 1. In contrast, Akiyama–Kaneko introduced a multiplicative analogue using fractional parts of the geometric progression $\alpha\beta^n$ [**AK21**]. In particular, when β is a Pisot number they prove closedness results for the spectrum; they also describe differences between the case where β is an integer and the case where β is a quadratic unit, including the existence of intervals and the structure of the first accumulation points and isolated points below them [**AK21, AK22**]. More recent work of Akiyama–Kamae–Kaneko extends formulas relating this multiplicative spectrum to symbolic dynamics to broader polynomial and recurrence-theoretic settings [**AKK25**].

(6) **Asymmetric and inhomogeneous approximation spectra**
This text focused on the usual Lagrange and Markov constants, but other spectra arise when one treats left and right approximations asymmetrically or adds inhomogeneous terms. Tornheim's asymmetric approximation is a classical example [**Tor55**]. For inhomogeneous minima of binary quadratic forms, the series of papers by Barnes and Swinnerton-Dyer is a standard classical reference [**BSD52a, BSD52b, BSD54**]. The minimum problems for binary quadratic forms and the continued-fraction descriptions developed in this text give a useful basis for understanding such variants. It is natural to ask whether the discrete values arising from generalized Markov numbers also appear in spectra other than the standard $\mathcal{L}$ and $\mathcal{M}$.

(7) **Frobenius's uniqueness conjecture and partial results**
Frobenius's uniqueness conjecture is one of the best-known open problems about Markov numbers. Aigner's book presents the conjecture together with Markov's theorem, fraction labels, and perfect matchings as a coherent story [**Aig13**]. The conjecture remains open, but it is known when the largest component is prime by work of Button, and for prime powers by results of Schmutz, Lang–Tan, and Zhang [**But98, LT07, Zha06**]. For the generalized uniqueness conjecture stated in Conjecture 8.6.7, the prime-power version in the case $k_1 = k_2 = k_3$ has been resolved by Gyoda–Maruyama [**GM23b**].

(8) **Order by fraction labels and Aigner-type conjectures**
Frobenius's fraction labels do not merely enumerate Markov numbers; they also provide coordinates for studying their order. Lee–Li–Rabideau–Schiffler gave precise inequalities that determine the order of Markov numbers from slopes and lattice data [**LLRS23**]. McShane gave a new proof of a related conjecture using convexity of length functions in hyperbolic geometry [**McS21**]. For generalized Markov numbers, analogous and generalized results in the case $k = k_1 = k_2 = k_3$ appear in work of Banaian and Banaian–Huang [**Ban25, BH26**].

(9) **Christoffel words and Sturmian words**
The correspondence between Markov numbers and reduced fractions is closely related to Christoffel words, which encode lattice segments, and to their non-periodic analogues, Sturmian words. Reutenauer made the correspondence between Christoffel words and Markov triples explicit and developed this subject systematically from a combinatorial viewpoint [**Reu09, Reu19**]. Cohn's matrix construction sends words to products in $SL(2,\mathbb{Z})$ and thereby links traces, continued fractions, and Markov numbers [**Coh55, Coh71**]. The generalized Cohn matrices in this text can be viewed as a generalization of this classical passage from words to matrices. A careful study of reversal, cyclic shift, and the classification of primitive words also clarifies the meaning of the matrix descriptions in Chapter 7.

(10) **Simple closed geodesics on the once-punctured torus**
Markov numbers are closely related to lengths of simple closed geodesics on the once-punctured torus. Cohn described Markov forms using geodesics on this torus [**Coh71**], and McShane–Rivin studied lengths of simple geodesics as a norm on homology [**MR95a, MR95b**]. In this viewpoint, fraction labels correspond to slopes of primitive lattice vectors in the universal cover, and Markov numbers record the associated geodesic

lengths. Recent work of Fisac translates the simple length spectrum into combinatorics of cyclic shift classes of integer sequences and gives a new formulation of the uniqueness conjecture [**Cam25**].

(11) **Markoff maps, Bowditch space, and McShane identities**
If Markov triples are allowed to take complex values and are regarded as functions on the trivalent tree, one obtains the theory of Markoff maps. Bowditch related Markoff triples to quasifuchsian representations of the once-punctured torus group and derived Bowditch conditions and variants of McShane identities [**Bow98**]. In this direction, the Markov equation is not merely an integer equation; it is a trace identity on the character variety of $SL(2, \mathbb{C})$-representations of the free group F_2. Although this text mainly treats integer-valued triples, the same tree structure and mutation operations also occur on complex character varieties.

(12) **Decorated Teichmüller space and λ-lengths**
In Penner's decorated Teichmüller theory, arcs on a surface are assigned positive real numbers called λ-lengths, and diagonal exchange in a quadrilateral is governed by the Ptolemy relation [**Pen87**]. This is one geometric origin of the modern principle that flips of triangulations correspond to mutations in cluster algebras. In higher Teichmüller theory, Fock–Goncharov introduced positive structures and cluster coordinates on moduli spaces of local systems [**FG06, FG09**]. The generalized Markov equations in this text are therefore connected not only to formal algebraic modifications, but also to positivity, Ptolemy-type relations, and the geometry of mutation.

(13) **Cluster algebras from surfaces and snake graph calculus**
Fomin–Shapiro–Thurston constructed the correspondence between tagged triangulations of bordered surfaces and seeds of cluster algebras [**FST08**]. Fomin–Thurston related this to the geometry of λ-lengths and interpreted cluster variables from surfaces as normalized λ-lengths [**FT18**]. Musiker–Schiffler–Williams expressed cluster variables from surfaces by perfect matchings of snake graphs [**MSW11**], and Canakci–Schiffler developed the relation between snake graph calculus and continued fractions [**ÇS13, ÇS18**]. The fence posets, skein relations, and GM distances in Chapter 6 can be understood naturally by comparing them with such perfect-matching formulas.

(14) **Further structure of generalized Cohn matrices**
The generalized Cohn matrices treated in this text realize generalized Markov numbers as matrix entries, but they also carry richer structure. In work of Gyoda–Maruyama–Sato, both generalized Cohn matrices and the parallel family called Markov–monodromy matrices are introduced as families of matrices in $SL(2, \mathbb{Z})$, and they recover the tree of positive integer solutions of the generalized Markov equation [**GMS24**]. In work of Banaian–Gyoda, these matrices are lifted to matrices with Laurent-polynomial entries, giving cluster structures on generalized Cohn and Markov–monodromy matrices [**BG25**]. Thus generalized Cohn matrices naturally connect the theory developed here with cluster algebras and the combinatorics of surfaces.

(15) **The place of the generalized discrete Markov spectrum**
The results of Chapter 8 show that the discrete values built from generalized Markov numbers belong to the Lagrange spectrum. Much remains unclear: how large these sets are inside $\mathcal{L}$, what their accumulation points are, and how the sets obtained from different coefficients (k_1, k_2, k_3) intersect. In the case $(0, 0, 0)$ one recovers the classical discrete part below 3, but for general coefficients the resulting values may also occur above 3. Thus the generalized discrete Markov spectrum can be viewed as a concrete source of values linking the classical discrete part and the transition region.

(16) **q-deformations, mirror deformations, and weighted perfect matchings**
Polynomial and Laurent-polynomial deformations of Markov numbers have been studied actively in recent years. Morier-Genoud–Ovsienko introduced q-rationals and q-continued fractions and related them to the Farey tree and triangulations [**MGO20**]. Kantarci Oguz gave a combinatorial model for q-deformed Markov numbers using directed posets and rank matrices [**Ogu25**]. Evans–Jouteur–Morier-Genoud–Ovsienko described

q-Markov numbers by q-deformed Cohn matrices and weighted perfect matchings of snake graphs [**EJMGO25**]. Bittmann–Jouteur–Kantarci Oguz–Molander–Yildirim introduced mirror Markov numbers and connected deformed Markov equations, mutations, and orbifold geometry [**BJKO$^+$26**]. Generalized Markov numbers may eventually fit into similar weighted or deformed frameworks.

(17) **Frieze patterns and Markov numbers**

Conway–Coxeter frieze patterns are closely related to cluster algebras of type A, triangulations, and Ptolemy relations. Propp explained the combinatorics of frieze patterns and Markov numbers through a model using perfect matchings, giving an intuitive explanation of positivity and the Laurent phenomenon [**Pro20**]. Morier-Genoud's survey on frieze patterns is also a useful entry point from classical friezes to modern cluster algebras [**MG15**]. Although the generalized Cohn matrices and fence posets of this text are not frieze patterns themselves, they share the same underlying features: Ptolemy-type relations and perfect matchings, or equivalently order ideals.

(18) **Markov–Hurwitz equations and higher-dimensional analogues**

The classical Markov equation has three variables, but higher-dimensional analogues such as

$$x_1^2 + x_2^2 + \cdots + x_n^2 = ax_1x_2\cdots x_n + k$$

have also been studied. Gamburd–Magee–Ronan obtained asymptotic formulas for the number of integer points when $n \geq 4$ [**GMR19**]. In higher dimensions the Vieta-jumping graph is no longer a simple trivalent tree, and questions about orbits of integer points, growth, and geometry of numbers become central. It is natural to ask whether generalized Markov equations can also be extended by increasing the number of variables, and whether any connection with spectra survives.

(19) **Markov equations over finite fields**

One may also study the Markov equation over finite fields $\mathbb{F}_p$. Then the Vieta involutions generate a graph on a finite set of solutions. Bourgain–Gamburd–Sarnak studied the action of Vieta involutions on congruence solutions of the Markov surface and gave applications to strong approximation and sieve theory [**BGS16**]. Chen further proved that, except for finitely many primes p, the Vieta involutions act transitively on the solution set of the Markov equation over $\mathbb{F}_p$ [**Che24**]. For generalized Markov equations it is natural to ask what connected components the congruence-solution graphs have and how strong approximation depends on (k_1, k_2, k_3); results in this direction appear in [**dCILM25, KN26**].

(20) **Markov-type K3 surfaces and arithmetic dynamics**

Markov-type equations also appear in the dynamics of K3 surfaces and character varieties. Fuchs–Litman–Silverman–Tran studied orbits of automorphism groups on Markov-type K3 surfaces, including orbit decompositions over finite fields and arithmetic-dynamical properties [**FLST24**]. In the classical Markov surface, Vieta involutions generate integer points; on K3 surfaces analogous involutions produce more complicated dynamics on higher-dimensional geometric objects. This viewpoint moves Markov-type equations from trees of integer solutions to actions of automorphism groups on algebraic varieties, and gives an important reference point for considering the algebro-geometric meaning of generalized Markov equations.

(21) **Symplectic geometry and $\mathbb{CP}^2$**

Markov triples also occur in exceptional bundles on $\mathbb{CP}^2$, weighted projective planes, and Lagrangian cell complexes. Classically, Rudakov used Markov numbers in the classification of exceptional bundles on $\mathbb{CP}^2$ [**Rud89**]. More recently, Evans–Smith studied the relation between Markov numbers and Lagrangian cell complexes in $\mathbb{CP}^2$, showing that Markov numbers arise naturally in symplectic geometry [**ES18**]. In this direction, the Markov equation appears away from Diophantine approximation, in contexts closer to surface degenerations, mirror symmetry, and Floer theory. It remains an open problem to identify what geometric objects are classified by the generalized Markov equations of this text, or what kind of mirror-side deformation they represent.

(22) **Toric geometry and Hirzebruch–Jung continued fractions**

Although this text mainly used regular continued fractions to study the Lagrange and Markov spectra, continued fractions also arise naturally in toric geometry. In particular, the Hirzebruch–Jung continued fraction

$$[b_1, \ldots, b_r]_- = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$$

describes the minimal resolution of a two-dimensional cyclic quotient singularity $\frac{1}{m}(1, q)$. Namely, if $\frac{m}{q} = [b_1, \ldots, b_r]_-$, then the exceptional curves form a chain whose self-intersection numbers are $-b_1, \ldots, -b_r$. For references, see [**Ful93**], [**CLS11**], and [**PP07**].

From this viewpoint Markov numbers are related to degenerations of algebraic surfaces and Wahl singularities. While the preceding item approached the same circle of ideas from the side of symplectic geometry, this is a birational-geometric viewpoint.

Urzua–Zuniga studied the birational-geometric structure of Markov numbers using the Hirzebruch–Jung continued fractions of Wahl singularities associated with Markov triples [**UZ23**].

Similar correspondences with cyclic quotient singularities also occur for generalized Markov numbers. For instance, k-Wahl chains are Hirzebruch–Jung continued fractions obtained inductively from $[k + 2]$ and have been studied as a class including cyclic quotient singularities arising from k-generalized Markov triples [**GMS24, Sat26**].

(23) **Growth laws**

Counting Markov numbers by size was studied classically by Zagier, who showed a logarithmic quadratic growth law for the number of Markov numbers below a given bound [**Zag82**]. Interpreted as counting simple closed geodesics on the once-punctured torus, this belongs to the same broad circle of ideas as Mirzakhani's theorem on the growth of simple closed geodesics [**Mir08**]. For generalized Markov numbers, one may simultaneously count depth in the tree, denominators of fraction labels, values of the numbers, and the associated spectral values; this may reveal growth laws different from the classical case. In computational experiments it is important to specify clearly which parameter is being counted.

(24) **Markov-type equations as mutation invariants of cluster algebras**

Cluster mutations often preserve polynomial invariants or positive integer solutions of Diophantine equations. Chen–Li classified sign-equivalence in mutation classes and gave applications to Markov-type equations [**CL25b**]. Recent preprints by Chen–Li, Bao–Li, and Chen–Jia study Markov-type equations from the viewpoints of mutation invariants, cluster symmetries, and tropicalization [**CL25a, BL25, CJ25**]. The generalized Markov equation in this text is another example of a mutation-preserved quantity read as an equation for positive integer solutions. Classifying which cluster-algebraic invariants give rise to good Diophantine equations is a natural way to extend the theory of generalized Markov numbers.

These topics show that the generalized discrete Markov spectrum studied in this text is not an isolated construction. It is related to several streams running from classical Diophantine approximation to cluster algebras, hyperbolic geometry, and arithmetic geometry. The purpose of this chapter is to indicate several paths through which readers can move further in these directions.

APPENDIX A

Proofs of Standard Facts Used in the Text

1. The Bolzano–Weierstrass Theorem

THEOREM A.1.1. *Every bounded real sequence $(x_n)_{n\geq 1}$ has a convergent subsequence.*

PROOF. Since (x_n) is bounded, there exist real numbers a_1, b_1 such that

$$a_1 \leq x_n \leq b_1 \qquad (n \in \mathbb{N})$$

for all n. Put $I_1 = [a_1, b_1]$.

Next, bisect I_1 at its midpoint. Then at least one of the two half-intervals contains infinitely many terms of the sequence (x_n). Indeed, if each of the two half-intervals contained only finitely many terms, then the total number of terms contained in I_1 would be finite, contradicting the fact that all terms of (x_n) lie in I_1.

Choose one of the half-intervals of I_1 that contains infinitely many terms of (x_n), and denote it by I_2. In the same way, once $I_k = [a_k, b_k]$ has been defined, bisect it and define $I_{k+1} = [a_{k+1}, b_{k+1}]$ to be one of the two halves that contains infinitely many terms of (x_n). In this way we obtain a sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

such that, for each k,

$$b_k - a_k = \frac{b_1 - a_1}{2^{k-1}}.$$

Moreover, each I_k contains infinitely many terms of (x_n).

We now choose a subsequence from these intervals. First choose one term x_{n_1} belonging to I_1. Since I_2 contains infinitely many terms, we may choose a term belonging to I_2 whose index is larger than n_1; call it x_{n_2}. Continuing in the same way, since I_k contains infinitely many terms, we may choose a term belonging to I_k whose index is larger than n_{k-1}; call it x_{n_k}. Then $n_1 < n_2 < \cdots$, and for each k we have $x_{n_k} \in I_k$. Thus (x_{n_k}) is a subsequence of (x_n).

It remains to show that this subsequence converges. Since the closed intervals are nested,

$$a_1 \leq a_2 \leq a_3 \leq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots.$$

The sequence (a_k) is bounded above, so by completeness of the real numbers the supremum

$$x := \sup\{a_k \mid k \in \mathbb{N}\}$$

exists.

For each fixed k, since $I_j \subset I_k$ for all $j \geq k$, and also $a_j \leq b_k$ for $j < k$ by the nesting and monotonicity above, b_k is an upper bound of the set $\{a_j\}$. Hence $x \leq b_k$. On the other hand, by definition we have $a_k \leq x$. Therefore

$$x \in [a_k, b_k] = I_k \qquad (k \in \mathbb{N}).$$

Consequently, for each k, both x_{n_k} and x belong to I_k. Hence

$$|x_{n_k} - x| \leq b_k - a_k = \frac{b_1 - a_1}{2^{k-1}}.$$

The right-hand side tends to 0 as $k \to \infty$, and therefore x_{n_k} converges to x. Thus (x_n) has a convergent subsequence. □

2. The Cayley–Hamilton Theorem

THEOREM A.2.1. *Let L be a field, and let $A \in M_n(L)$. Consider the characteristic polynomial of A,*

$$\chi_A(t) = \det(tE_n - A) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0.$$

Then

$$\chi_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 E_n = 0.$$

Here E_n denotes the $n \times n$ identity matrix.

PROOF. Take the adjugate matrix, that is, the transpose of the cofactor matrix, of $tE_n - A \in M_n(L[t])$. Then

$$\operatorname{adj}(tE_n - A)\,(tE_n - A) = \det(tE_n - A)\,E_n = \chi_A(t)E_n.$$

Each entry of $\operatorname{adj}(tE_n - A)$ is a polynomial of degree at most $n-1$. Hence there exist matrices $B_0, B_1, \ldots, B_{n-1} \in M_n(L)$ such that

$$\operatorname{adj}(tE_n - A) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0.$$

Substituting this into the identity above and expanding, we obtain

$$\begin{aligned}
&\big(B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0\big)(tE_n - A)\\
&= B_{n-1}t^n + (B_{n-2} - B_{n-1}A)t^{n-1} + \cdots + (B_0 - B_1 A)t - B_0 A\\
&= (t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0)E_n.
\end{aligned}$$

Therefore, by comparing coefficients, we obtain

$$B_{n-1} = E_n, \qquad B_{k-1} = B_k A + c_k E_n \quad (k = 1, 2, \ldots, n-1), \qquad -B_0 A = c_0 E_n.$$

Substituting these relations successively from the first one, we obtain

$$\begin{aligned}
B_{n-2} &= B_{n-1}A + c_{n-1}E_n = A + c_{n-1}E_n,\\
B_{n-3} &= B_{n-2}A + c_{n-2}E_n = A^2 + c_{n-1}A + c_{n-2}E_n,\\
&\vdots\\
B_0 &= A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_2 A + c_1 E_n.
\end{aligned}$$

Substituting this into $-B_0 A = c_0 E_n$, we obtain

$$-(A^n + c_{n-1}A^{n-1} + \cdots + c_2 A^2 + c_1 A) = c_0 E_n.$$

Hence

$$A^n + c_{n-1}A^{n-1} + \cdots + c_2 A^2 + c_1 A + c_0 E_n = 0.$$

This is precisely $\chi_A(A) = 0$. □

COROLLARY A.2.2. *Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M_2(K)$. Then*

$$A^2 - \operatorname{tr}(A)A + \det(A)E_2 = 0.$$

In particular,

$$A^2 - (a+d)A + (ad - bc)E_2 = 0.$$

PROOF. We have

$$\chi_A(t) = \det(tE_2 - A) = \det\begin{bmatrix} t-a & -b \\ -c & t-d \end{bmatrix} = (t-a)(t-d) - bc = t^2 - (a+d)t + (ad - bc).$$

Therefore, by the theorem,

$$\chi_A(A) = A^2 - (a+d)A + (ad - bc)E_2 = 0.$$

□

3. A Basis for the Space of Sequences Satisfying a Linear Recurrence

THEOREM A.3.1. *Let L be a field, and let $c_0, c_1, \dots, c_{n-1} \in L$. Suppose that a sequence $(a_m)_{m\geq 0}$ satisfies the homogeneous linear recurrence*

$$a_{m+n} + c_{n-1}a_{m+n-1} + \cdots + c_1 a_{m+1} + c_0 a_m = 0 \qquad (m \geq 0).$$

Assume that the characteristic polynomial $p(x) := x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ of this recurrence has n distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n$ over L. Then the L-vector space of all sequences satisfying this recurrence coincides with the span of $(\lambda_1^m)_{m\geq 0}$, $(\lambda_2^m)_{m\geq 0}$, $\dots$, $(\lambda_n^m)_{m\geq 0}$. Consequently, every solution sequence $(a_m)_{m\geq 0}$ is uniquely expressed as

$$a_m = \alpha_1\lambda_1^m + \alpha_2\lambda_2^m + \cdots + \alpha_n\lambda_n^m \qquad (m \geq 0).$$

PROOF. First, for each $i = 1, \dots, n$, the sequence $u^{(i)} = (\lambda_i^m)_{m\geq 0}$ satisfies the given recurrence. Indeed,

$$\lambda_i^{m+n} + c_{n-1}\lambda_i^{m+n-1} + \cdots + c_1\lambda_i^{m+1} + c_0\lambda_i^m = \lambda_i^m(\lambda_i^n + c_{n-1}\lambda_i^{n-1} + \cdots + c_1\lambda_i + c_0) = \lambda_i^m p(\lambda_i) = 0.$$

Thus $u^{(i)}$ is a solution, and therefore any linear combination of these sequences is also a solution. Hence

$$\operatorname{span}\{u^{(1)}, u^{(2)}, \dots, u^{(n)}\} \subseteq S,$$

where S denotes the space of all sequences satisfying this recurrence.

Next we prove the reverse inclusion. Take any $a = (a_m)_{m\geq 0} \in S$. We want to determine coefficients $\alpha_1, \dots, \alpha_n \in L$ by imposing

$$a_k = \alpha_1\lambda_1^k + \alpha_2\lambda_2^k + \cdots + \alpha_n\lambda_n^k \qquad (k = 0, 1, \dots, n-1).$$

These coefficients are obtained by solving the linear system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

The coefficient matrix on the left-hand side is a Vandermonde matrix, and its determinant is $\prod_{1\leq i<j\leq n}(\lambda_j - \lambda_i)$. Since $\lambda_1, \dots, \lambda_n$ are distinct, this determinant is nonzero. Therefore the system above has a unique solution

$$\alpha_1, \dots, \alpha_n.$$

Now put

$$b_m = \alpha_1\lambda_1^m + \alpha_2\lambda_2^m + \cdots + \alpha_n\lambda_n^m \qquad (m \geq 0).$$

Then $b = (b_m)_{m\geq 0}$ is a solution of the recurrence by what we have already shown. Moreover, it satisfies $a_k = b_k$ for $k = 0, 1, \dots, n-1$. Since both sequences satisfy the same recurrence, the first n terms determine all subsequent terms, and hence $(a_m)_{m\geq 0} = (b_m)_{m\geq 0}$. Therefore

$$S \subseteq \operatorname{span}\{u^{(1)}, u^{(2)}, \dots, u^{(n)}\}.$$

Combining the two inclusions gives

$$S = \operatorname{span}\{(\lambda_1^m)_{m\geq 0}, (\lambda_2^m)_{m\geq 0}, \dots, (\lambda_n^m)_{m\geq 0}\}.$$

The uniqueness follows immediately from the invertibility of the Vandermonde matrix. □

4. Density of Irrational Rotations

THEOREM A.4.1. *Let $\tau \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$\{n + \tau m \mid m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}/\mathbb{Z}$.

Proof. It is enough to prove that the set $\{m\tau \bmod 1 \mid m \in \mathbb{Z}\}$ is dense in $\mathbb{R}/\mathbb{Z}$. Let $\varepsilon > 0$. Choose N with $1/N < \varepsilon$. Among the $N+1$ fractional parts $0, \tau, 2\tau, \ldots, N\tau$, there are two whose distance modulo 1 is less than $1/N$. Hence there exists an integer q with $1 \leq q \leq N$ such that $0 < \|q\tau\| < \varepsilon$, where $\|x\|$ denotes the distance from x to the nearest integer. The inequality is strict on the left because τ is irrational. Put $\delta = \|q\tau\|$. Then the points $0, \delta, 2\delta, \ldots, \left\lfloor \frac{1}{\delta} \right\rfloor \delta$ are ε-dense in $\mathbb{R}/\mathbb{Z}$. Each of these points is congruent, up to sign, to a multiple of $q\tau$, and hence to a multiple of τ. Therefore $\{m\tau \bmod 1 \mid m \in \mathbb{Z}\}$ is ε-dense. Since $\varepsilon > 0$ was arbitrary, the desired density follows. $\square$

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